SOME SEPARATION AXIOMS IN BIGENERALIZED TOPOLOGICAL SPACES

PATTHARAPOHN TORTON

A thesis submitted in partial fulfillment of the requirements for

the degree of Master of Science in Mathematics Education

Mahasarakham University

April 2012

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The examining committee has unanimously approved this thesis, submitted by Mrs. Pattharapohn Torton, as a partial fulfillment of the requirements for the Master of Science degree in Mathematics Education, Mahasarakham University.

Examining Committee

....	Chairman
(Montri Thongmoon, Ph.D)	(Faculty graduate committee)
....	Committee
(Chokchai Viriyapong, Ph.D.)	(Advisor)
....	Committee
(Asst.Prof. Chawalit Boonpok, Ph.D.)	(Co - advisor)
....	Committee
(Ekjittra Srisarakham, Ph.D)	(Faculty graduate committee)
....	Committee
(Supunnee Sanpinij, Ph.D.)	(External expert)

Mahasarakham University has granted approval to accept this thesis as a partial fulfillment of the requirements for the Master of Science degree in Mathematics Education.

....	....
(Prof. La - orsri Sanoamuang, Ph.D.)	(Prof. Pradit. Terdtoon, Ph.D.)
Dean of the Faculty of Science	Dean of the Faculty of Graduate Studies
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ผู้วิจัย นางภัทรภร ต่อต้น
ปริญญา วิทยาศาสตรมหาบัณฑิต **สาขาวิชา** คณิตศาสตรศึกษา
กรรมการควบคุม อาจารย์ ดร.โชคชัย วิริยะพงษ์
ผู้ช่วยศาสตราจารย์ ดร.ชวลิต บุญปก
มหาวิทยาลัย มหาวิทยาลัยมหาสารคาม **ปีที่พิมพ์** 2555

บทคัดย่อ

ในงานวิจัยเล่มนี้ ผู้วิจัยได้สร้างสัจพจน์การแยกบนปริภูมิเชิงไบทอพอโลยีวางนัยทั่วไป ได้แก่ ปริภูมิ $\mu_{(m,n)}-T_0$ ปริภูมิ $\mu_{(m,n)}-T_1$ ปริภูมิ $\mu_{(m,n)}-T_2$ ปริภูมิ $\mu_{(m,n)}$ - เรกูลาร์ ปริภูมิ $\mu_{(m,n)}-T_3$ ปริภูมิ $\mu_{(m,n)}$ - นอร์มอล และ ปริภูมิ $\mu_{(m,n)}-T_4$ รวมถึงศึกษาสมบัติพื้นฐาน และความสัมพันธ์ของสัจพจน์การแยกระหว่างปริภูมิดังกล่าวข้างต้น

คำสำคัญ : ปริภูมิเชิงไบทอพอโลยีวางนัยทั่วไป; ปริภูมิ $\mu_{(m,n)}-T_0$; ปริภูมิ $\mu_{(m,n)}-T_1$; ปริภูมิ $\mu_{(m,n)}$ - เรกูลาร์; ปริภูมิ $\mu_{(m,n)}$ - นอร์มอล; ปริภูมิ $\mu_{(m,n)}-T_4$

TITLE SOME SEPARATION AXIOMS IN BIGENERALIZED TOPOLOGICAL SPACES

CANDIDATE Mrs. Pattharapohn Torton

DEGREE Master Degree of Science **MAJOR** Mathematics Education

ADVISORS Chokchai Viriyapong, Ph.D.

Asst. Prof. Chawalit Boonpok, Ph.D.

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ABSTRACT

In this research, the researcher constructed the separation axioms in bigeneralized to pological spaces including the $\mu_{(m,n)}-T_0$ space, $\mu_{(m,n)}-T_1$ space, $\mu_{(m,n)}-T_2$ space, $\mu_{(m,n)}$- regular space, $\mu_{(m,n)}-T_3$ space, $\mu_{(m,n)}$- normal space and $\mu_{(m,n)}-T_4$ space and studied some basic properties of the axioms as well as relationships among these spaces.

Keywords : bigeneralized topological space; $\mu_{(m,n)}-T_0$ space; $\mu_{(m,n)}-T_1$ space; $\mu_{(m,n)}$- regular space; $\mu_{(m,n)}$- normal space; $\mu_{(m,n)}-T_4$ space

CONTENTS

CHAPTER 1

INTRODUCTION

1.1 Background

General topology is important in many fields applied sciences as well as branches of mathematics. In reality it is used in data mining, computational topology for geometric design and modecular design, computer – aided design, computer – aided geometric design, digital topology, information system, paricle physis and quantum physics etc.

The theory of generalized topological space, which was introduced by Császár [3], is one of the most important developments of general topology in recent years. Later, he [4] studied some the simplest separation axioms by replacing open sets by an arbitrary family of subsets of a topological space. In 2010, Roy [12] introduced the concepts of the separation axioms R_0 and R_1 in generalized topological spaces. Furthermore, he gave some characterizations of such them. In 2011, The weak separation axiom, including $T_0, T_1, T_2, R_0, R_1, D_0, D_1$ and D_2 in generalized topological space were studies by SARSAK [14]. Furthermore, he investigated some characterizations of the axioms as well as the relationships among these axioms. Min [9] introduced the concepts of relative separation axioms in generalized topological spaces and investigate properties for such notions, in particular, the product of T_2-spaces is T_2- spaces. In the same time, GE Xun and GE Ying [15] gave some charactelizations of some separation axioms in generalized topological space. In 2010, Boonpok [1] introduced the concept of bigeneralized topological spaces and studied (m,n)- closed sets and (m,n)- open sets in bigeneralized topological spaces. In [10], Min introduced the notion of almost regular space in bigeneralized topological spaces. Furthermore, he [11] gave the concept of regular spaces in bigeneralized topological spaces.

According to the prior studies as mentioned above, I am interested in some separation axioms in bigeneralized topological spaces.

The thesis is divided into five chapters. The first chapter is formed by an introduction which contains some historical remarks concerning the research specialization. We also explain our motivations and outline the goals of the thesis here. In the second chapter, we give some definitions, notations and some known theorems that will be used in the later chapters. In the third chapter, we give some definitions, notations and some interesting propositions of $\mu_{(m,n)} - T_0$ spaces, $\mu_{(m,n)} - T_1$ spaces and $\mu_{(m,n)} - T_2$ spaces. We derived some of their properties. In the forth chapter, we give some definitions, notations and some known theorems of $\mu_{(m,n)}$- regular spaces and $\mu_{(m,n)}$- normal spaces and their characterizations. In the last chapter, we make conclusions of the obtained results and also outline the direction of the further research

1.2 Objectives of the research

The purpose of the research are:

1.2.1 To construct and study some separation axioms between two distinct points in bigeneralized topological spaces.

1.2.2 To construct and study some separation axioms between closed sets and the points outside this closed set in bigeneralized topological spaces.

1.2.3 To construct and study some separation axioms between two disjoints closed sets in bigeneralized topological spaces.

1.3 Research methodology

The research procedure of thesis consists of the following steps:

1.3.1 Criticism and possible extensions of the literature review.

1.3.2 Doing research to investigate the main results.

1.3.3 Applying the results from 1.3.1 and 1.3.2 to the main results.

1.3.4 Making the conclusions and recommendations.

1.4 Scope of the study

The scope of the study are:

1.4.1 Constructing some separation axioms between two distinct points in bigeneralized topological spaces.

1.4.2 Constructing some separation axioms between closed sets and the points outside this closed set in bigeneralized topological spaces.

1.4.3 To construct and study some separation axioms between two disjoints closed sets in bigeneralized topological spaces.

CHAPTER 2

PRELIMINARIES

This chapter includes definitions, notations and some known facts which are used throughout the thesis.

2.1 Generalized Topological Spaces

In this section, we gave some definitions, notations and known propositions of generalized topological spaces that will be used in the next chapter, as follows:

Definition 2.1.1 [3] Let X be a nonempty set and μ be a collection of subsets of X. Then μ is called a *generalized topology* (briefly GT) on X if and only if $\phi \in \mu$ and $G_i \in \mu$ for $i \in I \neq \phi$ implies $G = \bigcup_{i \in I} G_i \in \mu$. We call the pair (X,μ) a *generalized topological space* (briefly GTS). The elements of μ are called μ - *open sets* and the complements of μ- open sets are called μ - *closed sets*. A generalized topological space (X,μ) is said to be *strong* [15] if $X \in \mu$.

Definition 2.1.2 [3] Let X be a nonempty set and μ be a generalized topology on X and $A \subseteq X$.

The closure of a subset A in a generalized topological space (X,μ), denoted by $c_\mu(A)$, as follows

$$c_\mu(A) = \cap\{F \mid A \subseteq F, X - F \in \mu\}.$$

The interior of a subset A in a generalized topological space (X,μ), denoted by $i_\mu(A)$, as follows

$$i_\mu(A) = \cup\{G \mid G \subseteq A, G \in \mu\}.$$

Lemma 2.1.3 [7] Let (X,μ) be a generalized topological space and let $A,B \subseteq X$. Then

(1) $c_\mu(X-A)=X-i_\mu(A)$ and $i_\mu(X-A)=X-c_\mu(A)$.

(2) If $X-A\in\mu$, then $c_\mu(A)=A$ and if $A\in\mu$ then $i_\mu(A)=A$.

(3) If $A\subseteq B$, then $c_\mu(A)\subseteq c_\mu(B)$ and $i_\mu(A)\subseteq i_\mu(B)$.

(4) $A\subseteq c_\mu(A)$ and $i_\mu(A)\subseteq A$.

(5) $c_\mu(c_\mu(A))=c_\mu(A)$ and $i_\mu(i_\mu(A))=i_\mu(A)$.

Lemma 2.1.4 [8] Let (X,μ) be a generalized topological space and $A\subseteq X$. Then

(1) $x\in i_\mu(A)$ if and only if there exists $V\in\mu$ such that $x\in V\subseteq A$;

(2) $x\in c_\mu(A)$ if and only if $V\cap A\neq\phi$ for every μ-open set V containing x.

Definition 2.1.5 [7] A space (X,μ) is called $\mu-T_0$ if for any two distinct points of X, there is a μ- open set of X which contain one but not the other.

Example 2.1.6 [7] Let $X=\{a, b, c\}$ and $\mu=\{\phi,\{a\},\{a,b\}\}$.

We see that

- $\{a\}$ is μ- open set such that $a\in\{a\}$ but $b,c\notin\{a\}$;
- $\{a,b\}$ is μ- open set such that $b\in\{a,b\}$ but $c\notin\{a,b\}$.

Thus (X,μ) is $\mu-T_0$.

Definition 2.1.7 [7] A space (X,μ) is called $\mu-T_1$ if for any two distinct points x and y of X, there exist μ- open sets U and V of X such that $x\in U$ but $y\notin U$ and $y\in V$ but $x\notin V$.

Example 2.1.8 [7] Let $X=\{a, b, c\}$ and $\mu=\{X,\phi,\{a,b\},\{a,c\},\{b,c\}\}$.

We see that

- $\{a, c\}$ and $\{b, c\}$ are μ- open such that $a\in\{a, c\}$, but $b\notin\{a, c\}$ and $b\in\{b, c\}$ but $a\notin\{b, c\}$,
- $\{a, b\}$ and $\{a, c\}$ are μ- open such that $b\in\{a,b\}$ but $c\notin\{a,b\}$ and $c\in\{a, c\}$ but $b\notin\{a, c\}$,

- $\{a, b\}$ and $\{b, c\}$ are μ- open such that $a \in \{a,b\}$ but $c \notin \{a,b\}$ and $c \in \{b, c\}$ but $a \notin \{b, c\}$.

Thus (X,μ) is $\mu - T_1$.

Definition 2.1.9 [7] A space (X,μ) is called $\mu - T_2$ if for any two distinct points x and y of X, there exist disjoint μ- open sets U and V of X such that $x \in U$, $y \in V$ and $U \cap V = \phi$.

Theorem 2.1.10 [7] A space (X,μ) is $\mu - T_0$ if and only if for each pair of distinct points x, y of X, $c_\mu(\{x\}) \neq c_\mu(\{y\})$.

Theorem 2.1.11 [7] A space (X,μ) is $\mu - T_1$ if and only if the singleton of X are μ-closed.

Definition 2.1.12 [7] A space (X,μ) is said to be a $\mu - R_0$ *space* if every μ- open set contains the μ- closure of each of its singletons.

Example 2.1.13 [7] Let $X = \{a, b, c\}$ and $\mu = \{\phi, \{a,b\}, \{c\}, X\}$.

Since $c_\mu(\{a\}) = c_\mu(\{b\}) = \{a,b\} \subseteq \{a,b\}$ and $c_\mu(\{c\}) = \{c\} \subseteq \{c\}$, (X,μ) is $\mu - R_0$.

Definition 2.1.14 [7] A space (X,μ) is said to be $\mu - R_1$ if for any $x, y \in X$ with $c_\mu(\{x\}) \neq c_\mu(\{y\})$, there exist disjoint μ- open sets U and V such that $c_\mu(\{x\}) \subseteq U$ and $c_\mu(\{y\}) \subseteq V$.

Example 2.1.15 [7] Let $X = \{a, b, c\}$ and $\mu = \{\phi, \{a,b\}, \{c\}, X\}$.

Since $c_\mu(\{a\}) = c_\mu(\{b\}) = \{a,b\} \subseteq \{a,b\}$ and $c_\mu(\{c\}) = \{c\} \subseteq \{c\}$, (X,μ) is $\mu - R_1$.

Theorem 2.1.16 [7] If (X,μ) is $\mu - R_1$, then (X,μ) is $\mu - R_0$.

Definition 2.1.17 [7] A space (X,μ) is said to be μ - *symmetric* if for each $x, y \in X$, $x \in c_\mu(\{y\})$ implies $y \in c_\mu(\{x\})$.

Theorem 2.1.18 [7] A space (X,μ) is $\mu-R_0$ if and only if (X,μ) is μ- symmetric.

Theorem 2.1.19 [7] A space (X,μ) is $\mu-T_1$ if and only if (X,μ) is $\mu-T_0$ and $\mu-R_0$.

Corolary 2.1.20 [7] For a $\mu-R_0$ space (X,μ), the following are equivalent:

(1) (X,μ) is $\mu-T_0$,

(2) (X,μ) is $\mu-T_1$.

Theorem 2.1.21 [7] For a space (X,μ), the following are equivalent:

(1) (X,μ) is $\mu-T_2$,

(2) (X,μ) is $\mu-T_1$ and $\mu-R_1$,

(3) (X,μ) is $\mu-T_0$ and $\mu-R_1$.

Corolary 2.1.22 [7] For a $\mu-R_1$ space (X,μ), the following are equivalent:

(1) (X,μ) is $\mu-T_2$,

(2) (X,μ) is $\mu-T_1$,

(3) (X,μ) is $\mu-T_0$.

Definition 2.1.23 [15] Let X be a strong generalized topological space. X is called *$\mu-T_3$ space* if and only if for all $x\in X$ and F is μ- closed set with $x\notin F$, then there are μ- open U and V with $x\in U$, $F\subseteq V$ and $U\cap V=\phi$.

Theorem 2.1.24 [15] The following are equivalent for a space (X,μ).

1. X is $\mu-T_3$ space.

2. If $x\notin F$ with F is μ- closed, then there are $U,V\in\mu$ with $x\in U, F\subseteq V$ and $c_\mu(U)\cap V=\phi$.

3. If $x\notin F$ with F is μ- closed, then there is $U\in\mu$ with $x\in U$ and $c_\mu(U)\cap F=\phi$.

4. If $x\in X$ and $U\in\mu$ such that $x\in U$, then there is $V\in\mu$ with $x\in V\subseteq c_\mu(V)\subseteq U$.

5. $F=\cap\{c_\mu(U): F\subseteq U\in\mu\}$ for each μ- closed subset F of X.

Definition 2.1.25 [15] Let (X,μ) be a strong generalized topological space. Then X is called *$\mu-T_4$ space* if F_1 and F_2 are μ- closed and $F_1 \cap F_2 = \phi$, then there are $U, V \in \mu$ with $F_1 \subseteq U, F_2 \subseteq V$ and $U \cap V = \phi$.

Theorem 2.1.26 [15] The following are equivalent for a space (X,μ).

1. X is a $\mu-T_4$ space.
2. If F_1, F_2 are μ- closed, $F_1 \cap F_2 = \phi$, then there are $U, V \in \mu$ such that $F_1 \subseteq U, F_2 \subseteq V$ and $c_\mu(U) \cap V = \phi$.
3. If F_1, F_2 are μ- closed, $F_1 \cap F_2 = \phi$, then there is $U \in \mu$ such that $F_1 \subseteq U$ and $c_\mu(U) \cap F_2 = \phi$.
4. If F is μ- closed and $F \subseteq U \in \mu$, then there is $V \in \mu$ such that $F \subseteq V \subseteq c_\mu(V) \subseteq U$.

Definition 2.1.27 [14] Let A be a subset of a space (X,μ). Then A is called *$g\mu$- closed* if $c_\mu(A) \subseteq U$ whenever $A \subseteq U$ and $U \in \mu$. A is called $g\mu$- open if $X - A$ is $g\mu$- closed, or equivalently, $F \subseteq i_\mu(A)$ whenever $F \subseteq A$ and F is μ- closed.

2.2 Bigeneralized Topological Space

Definition 2.2.1 [1] Let X be a nonempty set and let μ_1 and μ_2 be generalized topologies on X. A triple (X, μ_1, μ_2) is said to be a *bigeneralized topological space* (briefly BGTS). For any subset A of X closure of A and interior of A with respect to μ_m are denoted by $c_{\mu_m}(A)$ and $i_{\mu_m}(A)$, respectively, for m=1 or 2.

Example 2.2.2 [1] Let $X = \{a,b,c,d\}$, $\mu_1 = \{\phi, \{a\}, \{b\}, \{a, b\}\}$ and $\mu_2 = \{\phi, \{c\}, \{a, c\}\}$. Then (X, μ_1, μ_2) is bigeneralized topological space. Let $A = \{b, c\}$. Thus

$$c_{\mu_1}(A) = c_{\mu_1}(\{b, c\}) = \{b, c, d\}, \qquad i_{\mu_1}(A) = i_{\mu_1}(\{b, c\}) = \{b\},$$

$$c_{\mu_2}(A) = c_{\mu_2}(\{b, c\}) = X, \qquad i_{\mu_2}(A) = i_{\mu_2}(\{b, c\}) = \{c\}.$$

Definition 2.2.3 [1] Let (X, μ_1, μ_2) be a bigeneralized topological space. A subset A of X is called (m,n)- *closed set* if $c_{\mu_m}(c_{\mu_n}(A)) = A$ where $m, n = 1, 2$ and $m \neq n$. The complements of (m,n)- closed set are call (m,n)- open sets.

Lemma 2.2.4 [1] Let (X, μ_1, μ_2) be a bigeneralized topological space and $A \subseteq X$. Then A is a (m,n)- closed set if and only if A is a μ_1- closed set and a μ_2- closed.

Lemma 2.2.5 [1] Let (X, μ_1, μ_2) be a bigeneralized topological space. If A and B are (m,n) - closed then $A \cap B$ is a (m,n)- closed set.

Remark 2.2.6 [1] The union of two (m,n)- closed sets is not a (m,n)- closed set in general as can be seen from the following example.

Example 2.2.7 [1] Let (X, μ_1, μ_2) be a bigeneralized topological space with $X = \{a, b, c, d\}$, $\mu_1 = \{\phi, \{a, c, d\}, \{b, c, d\}, X\}$ and $\mu_2 = \{\phi, \{a, c, d\}, \{b, c, d\}, X\}$.
We see that $X, \{b\}, \{a\}$ and ϕ are μ_1- closed and μ_2- closed. By Lemma 2.2.5, $X, \{b\}, \{a\}$ and ϕ are (m,n)- closed sets, Where $m,n = 1,2$ and $m \neq n$. Then $\{a\}$ and $\{b\}$ are (m,n)- closed, but $\{a\} \cup \{b\} = \{a, b\}$ is not a (m,n)- closed set.

Lemma 2.2.8 [1] Let (X, μ_1, μ_2) be a bigeneralized topological space and $A \subseteq X$. Then A is a (m,n)- open set if and only if $i_{\mu_m}(i_{\mu_n}(A)) = A$.

Lemma 2.2.9 [1] Let (X, μ_1, μ_2) be a bigeneralized topological space If A and B is (m,n)- open set then $A \cup B$ is (m,n)- open set.

Remark 2.2.10 [1] The intersection of two (m,n)- open sets is not a (m,n)- open set in general as can be seen from the following example.

Example 2.2.11 [1] Let (X, μ_1, μ_2) be a bigeneralized topological space with $X = \{a, b, c, d\}$, $\mu_1 = \{\phi, \{a, b\}, \{b, c\}, \{a, b, c\}, X\}$ and $\mu_2 = \{\phi, \{a, b\}, \{b, c\}, \{a, b, c\}, X\}$.
We see that $\{a, b\}$ and $\{b, c\}$ are $(1,2)$- open. But $\{a, b\} \cap \{b, c\} = \{b\}$ is not $(1,2)$- open.

CHAPTER 3

WEAK SEPARATION AXIOMS IN BIGENERALIZED TOPOLOGICAL SPACES

In this chapter, we introduce the notions of weak separation axioms in bigeneralized topological space. Next, we study some properties of them.

Throughout this chapter, we let $m, n \in \{1, 2\}$ where $m \neq n$.

3.1 $\mu_{(m,n)} - T_0$ spaces and $\mu_{(m,n)} - T_1$ spaces

In this section, we give the definition of $\mu_{(m,n)} - T_0$ space and $\mu_{(m,n)} - T_1$ space. After that, we give characterize these spaces.

Definition 3.1.1 A bigeneralized topological space (X, μ_1, μ_2) is called $\mu_{(m,n)} - T_0$ if for any pair of distinct points of X, there exists a μ_m- open set or a μ_n- open set contain one of the points but not the other. That is, (X, μ_1, μ_2) is $\mu_{(m,n)} - T_0$ if and only if for any $x, y \in X$ with $x \neq y$, there exists a subset U of X such that U is μ_m- open or μ_n- open and $x \in U$ but $y \notin U$ or $y \in U$ but $x \notin U$.

Example 3.1.2 Let $X = \{a, b, c\}$. We define two generalized topologies on X as follows: $\mu_1 = \{\phi, \{a, b\}\}$ and $\mu_2 = \{\phi, \{b, c\}\}$. Then (X, μ_1, μ_2) is $\mu_{(1,2)} - T_0$ and $\mu_{(2,1)} - T_0$.

Theorem 3.1.3 Let (X, μ_1, μ_2) be a bigeneralized topological space. Then (X, μ_1, μ_2) is $\mu_{(m,n)} - T_0$ if and only if for each pair of distinct points x, y of X, $c_{\mu_m}(\{x\}) \neq c_{\mu_m}(\{y\})$ or $c_{\mu_n}(\{x\}) \neq c_{\mu_n}(\{y\})$.

Proof. $(\Rightarrow)$ Assume that (X, μ_1, μ_2) is $\mu_{(m,n)} - T_0$. Let $x, y \in X$ with $x \neq y$. Then there exists $U \subseteq X$ such that U is μ_m- open or μ_n- open, $x \in U$ but $y \notin U$ or $y \in U$ but $x \notin U$.

Without loss of generality, we assume that $x \in U$ but $y \notin U$. If U is μ_m- open, then $x \notin c_{\mu_m}(\{y\})$, and so $c_{\mu_m}(\{x\}) \neq c_{\mu_m}(\{y\})$. If U is μ_n- open, then $x \notin c_{\mu_n}(\{y\})$, and so $c_{\mu_n}(\{x\}) \neq c_{\mu_n}(\{y\})$. Hence, $c_{\mu_m}(\{x\}) \neq c_{\mu_m}(\{y\})$ or $c_{\mu_n}(\{x\}) \neq c_{\mu_n}(\{y\})$.

($\Leftarrow$) Assume that $c_{\mu_m}(\{x\}) \neq c_{\mu_m}(\{y\})$ or $c_{\mu_n}(\{x\}) \neq c_{\mu_n}(\{y\})$ for each $x, y \in X$ with $x \neq y$. We will show that (X, μ_1, μ_2) is $\mu_{(m,n)} - T_0$. Let $x, y \in X$ with $x \neq y$. By assumption, $c_{\mu_m}(\{x\}) \neq c_{\mu_m}(\{y\})$ or $c_{\mu_n}(\{x\}) \neq c_{\mu_n}(\{y\})$. If $c_{\mu_m}(\{x\}) \neq c_{\mu_m}(\{y\})$, then, without loss of generality, we assume that $c_{\mu_m}(\{x\}) \not\subset c_{\mu_m}(\{y\})$. Hence $x \notin c_{\mu_m}(\{y\})$. Set $U = X - c_{\mu_m}(\{y\})$. Then U is a μ_m- open subset of X and $x \in U$ but $y \notin U$. Similarly, we can prove that if $c_{\mu_n}(\{x\}) \neq c_{\mu_n}(\{y\})$, then there exists a μ_n- open subset U of X which contains one but not the other. Therefore, (X, μ_1, μ_2) is $\mu_{(m,n)} - T_0$.

Remark 3.1.4 Let (X, μ_1, μ_2) be a bigeneralized topological space**.** From the above theorem, (X, μ_1, μ_2) is $\mu_{(m,n)} - T_0$ if and only if (X, μ_1, μ_2) is $\mu_{(n,m)} - T_0$.

Proposition 3.1.5 Let (X, μ_1, μ_2) be a bigeneralized topological space. If (X, μ_m) is $\mu_m - T_0$ or (X, μ_n) is $\mu_n - T_0$, then (X, μ_1, μ_2) is $\mu_{(m,n)} - T_0$.

Proof. It follows from theorem 3.1.3 and Theorem 2.1.9.

Example 3.1.6 Let $X = \{a, b, c\}$. We define two generalized topologies on X as follows: $\mu_1 = \{\phi, \{a, b\}, X\}$ and $\mu_2 = \{\phi, \{a, c\}, X\}$. Then (X, μ_1, μ_2) is $\mu_{(1,2)} - T_0$ but (X, μ_1) is not $\mu_1 - T_0$ and (X, μ_2) is not $\mu_2 - T_0$.

Definition 3.1.7 A bigeneralized topological space (X, μ_1, μ_2) is said to be $\mu_{(m,n)} - T_1$ if for any $x, y \in X$ with $x \neq y$, there exist a μ_m- open set U and a μ_n- open set V such that $x \in U$ but $y \notin U$ and $x \notin V$ and $y \in V$.

Example 3.1.8 Let $X = \{a, b, c\}$. We define two generalized topologies on X as follows: $\mu_1 = \{\phi, \{a\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$ and $\mu_2 = \{\phi, \{a, b\}, \{a, c\}, \{b, c\}, X\}$. Then (X, μ_1, μ_2) is $\mu_{(1,2)}$-T_1 and $\mu_{(2,1)}$-T_1.

Example 3.1.9 Let $X = \{a, b\}$. We define two generalized topologies on X as follows: $\mu_1 = \{\phi, \{a\}\}$ and $\mu_2 = \{\phi, \{b\}\}$.

Then (X, μ_1, μ_2) is not $\mu_{(1,2)}$-T_1 because if $b \neq a$ but there is no μ_1- open set containing b but not a.

Remark 3.1.10 Let (X, μ_1, μ_2) be a bigeneralized topological space. Then (X, μ_1, μ_2) Is $\mu_{(m,n)}$-T_1 if and only if for any $x, y \in X$ with $x \neq y$, there exist μ_m-open sets U_1, U_2 and μ_n-open sets V_1, V_2 such that

1. $x \in U_1$ but $y \notin U_1$ and $y \in V_1$ but $x \notin V_1$,
2. $y \in U_2$ but $x \notin U_2$ and $x \in V_2$ but $y \notin V_2$.

Remark 3.1.11 It is clear that every $\mu_{(m,n)} - T_1$ space is a $\mu_{(m,n)} - T_0$ space. But a $\mu_{(m,n)} - T_0$ space is not a $\mu_{(m,n)} - T_1$ space, in general, as the following example.

Example 3.1.12 Let $X = \{a, b, c\}$. We define two generalized topologies on X as follows: $\mu_1 = \{\phi, \{a\}, \{a, b\}, X\}$ and $\mu_2 = \{\phi, \{b\}, \{a, c\}, X\}$.
Then (X, μ_1, μ_2) is a $\mu_{(1,2)}$-T_0 space but it is not a $\mu_{(1,2)}$-T_1 space.

Theorem 3.1.13 Let (X, μ_1, μ_2) be a bigeneralized topological space. Then (X, μ_1, μ_2) is $\mu_{(m,n)} - T_1$ if and only if $\{x\}$ is μ_m- closed set and μ_n- closed set, for all $x \in X$.

Proof. $(\Rightarrow)$ Assume that (X, μ_1, μ_2) is $\mu_{(m,n)} - T_1$. Let $x \in X$. By assumption, for each $y \in X$ such that $y \neq x$, there exist a μ_m- open set U_y and a μ_n- open set V_y such that $y \in U_y$ but $x \notin U_y$ and $y \in V_y$ but $x \notin V_y$. Then $X - \{x\} = \bigcup_{y \in X - \{x\}} U_y$ is a μ_m- open set and $X - \{x\} = \bigcup_{y \in X - \{x\}} V_y$ is a μ_n- open set. Hence $\{x\}$ is μ_m- closed and μ_n- closed.

$(\Leftarrow)$ Assume that $\{x\}$ is μ_m- closed and μ_n- closed, for all $x \in X$. To show that (X, μ_1, μ_2) is $\mu_{(m,n)} - T_1$, let $x, y \in X$ with $x \neq y$. By assumption, we obtain that $\{x\}$ is μ_n- closed and $\{y\}$ is μ_m- closed. set $U = X - \{y\}$ and $V = X - \{x\}$. Thus U is μ_m- open and V is μ_n- open. Moreover, $x \in U$ but $y \notin U$ and $y \in V$ but $x \notin V$. Then (X, μ_1, μ_2) is $\mu_{(m,n)} - T_1$.

Remark 3.1.14 Let (X, μ_1, μ_2) be a bigeneralized topological space. From the previous theorem, it is clear that (X, μ_1, μ_2) is $\mu_{(m,n)} - T_1$ if and only if $\mu_{(n,m)} - T_1$.

Definition 3.1.15 A bigeneralized topological space (X, μ_1, μ_2) is said to be a $\mu_{(m,n)} - R_0$ *space* if every μ_m- open set contains the μ_n- closure of each of its singleton. That is, (X, μ_1, μ_2) is $\mu_{(m,n)} - R_0$ if and only if for all μ_m- open set U, if $x \in U$ then $c_{\mu_n}(\{x\}) \subseteq U$.

Example 3.1.16 Let $X = \{a, b, c\}$. We define two generalized topologies on X as follows: $\mu_1 = \{\phi, \{a,b\}, \{a,c\}, X\}$ and $\mu_2 = \{\phi, \{a,b\}, \{a,c\}, \{b,c\}, X\}$.
Then (X, μ_1, μ_2) is $\mu_{(1,2)} - R_0$ but it is not $\mu_{(2,1)} - R_0$.

Definition 3.1.17 A bigeneralized topological space (X, μ_1, μ_2) is said to be *pairwise* R_0 if (X, μ_1, μ_2) is $\mu_{(1,2)} - R_0$ and $\mu_{(2,1)} - R_0$.

Theorem 3.1.18 Let (X, μ_1, μ_2) be a bigeneralized topological space. Then (X, μ_1, μ_2) is a $\mu_{(m,n)} - R_0$ *space* if and only if for every μ_m- closed subset F, if $x \notin F$, then $c_{\mu_n}(\{x\}) \cap F = \phi$.

Proof. $(\Rightarrow)$ Assume that (X, μ_1, μ_2) is $\mu_{(m,n)} - R_0$. Let F be a μ_m- closed subset of X. Assume that $x \notin F$. Then $x \in X - F$ and $X - F$ is μ_m- open. Since (X, μ_1, μ_2) is $\mu_{(m,n)} - R_0$, $c_{\mu_n}(\{x\}) \subseteq X - F$. Hence $c_{\mu_n}(\{x\}) \cap F = \phi$.

$(\Leftarrow)$ Let U be a μ_m- open subset of X. Assume that $x \in U$. Then $x \notin X - U$ and $X - U$ is μ_m- closed. By assumption, $c_{\mu_n}(\{x\}) \cap (X - U) = \phi$. Then $c_{\mu_n}(\{x\}) \subseteq U$. Therefore, (X, μ_1, μ_2) is $\mu_{(m,n)} - R_0$.

Definition 3.1.19 A bigeneralized topological space (X, μ_1, μ_2) is said to be a $\mu_{(m,n)} - R_1$ space if for any $x, y \in X$ with $c_{\mu_m}(\{x\}) \neq c_{\mu_n}(\{y\})$, there exist μ_m- open set U and μ_n- open set V such that $c_{\mu_m}(\{x\}) \subseteq V$, $c_{\mu_n}(\{y\}) \subseteq U$ and $U \cap V = \phi$.

Remark 3.1.20 From the above definition, we obtain that (X, μ_1, μ_2) is $\mu_{(m,n)} - R_1$ if and only if (X, μ_1, μ_2) is $\mu_{(n,m)} - R_1$.

Theorem 3.1.21 Let (X, μ_1, μ_2) be a bigeneralized topological space. If (X, μ_1, μ_2) is $\mu_{(m,n)} - R_1$, then (X, μ_1, μ_2) is $\mu_{(m,n)} - R_0$.

Proof. Assume that (X, μ_1, μ_2) is $\mu_{(m,n)} - R_1$. We will show that (X, μ_1, μ_2) is $\mu_{(m,n)} - R_0$.

Let U be a μ_m- open set and $x \in U$. To show that $c_{\mu_n}(\{x\}) \subseteq U$, let $y \notin U$. Then $U \cap \{y\} = \phi$, implies that $x \notin c_{\mu_m}(\{y\})$. Hence $c_{\mu_n}(\{x\}) \neq c_{\mu_m}(\{y\})$. Since (X, μ_1, μ_2) is $\mu_{(m,n)} - R_1$, there exist μ_m- open set U_0 and μ_n- open set V_0 such that $c_{\mu_n}(\{x\}) \subseteq U_0$, $c_{\mu_m}(\{y\}) \subseteq V_0$ and $U_0 \cap V_0 = \phi$. Thus $y \in V_0$ and $\{x\} \cap V_0 = \phi$. Hence $y \notin c_{\mu_n}(\{x\})$. Therefore, $c_{\mu_n}(\{x\}) \subseteq U$.

Remark 3.1.22 In general, a $\mu_{(m,n)}$- R_0 space is not a $\mu_{(m,n)}$- R_1 space as can be seen from the following example.

Example 3.1.23 Let $X = \{a, b, c\}$. We define two generalized topologies on X as follow: $\mu_1 = \{\phi, \{a,b\}, \{a,c\}, X\}$ and $\mu_2 = \{\phi, X\}$. Then (X, μ_1, μ_2) is $\mu_{(m,n)}$- R_0 space but it is not a $\mu_{(m,n)}$- R_1.

Definition 3.1.24 A bigeneralized topological space (X, μ_1, μ_2) is said to be a $\mu_{(m,n)}$-*symmetric* if for each $x, y \in X$, $x \in c_{\mu_n}(\{y\})$ implies $y \in c_{\mu_m}(\{x\})$.

Theorem 3.1.25 Let (X, μ_1, μ_2) be a bigeneralized topological space. Then (X, μ_1, μ_2) is $\mu_{(m,n)} - R_0$ if and only if (X, μ_1, μ_2) is $\mu_{(m,n)}$ – symmetric.

Proof. $(\Rightarrow)$ Assume that (X, μ_1, μ_2) is $\mu_{(m,n)} - R_0$. Let x, y be elements of X such that $x \in c_{\mu_n}(\{y\})$ and let U be a μ_m- open set such that $y \in U$. Since (X, μ_1, μ_2) is $\mu_{(m,n)} - R_0$ and $y \in U$, $c_{\mu_n}(\{y\}) \subseteq U$. Hence $\{x\} \cap U \neq \phi$, and so $y \in c_{\mu_m}(\{x\})$.

$(\Leftarrow)$ Assume that (X, μ_1, μ_2) is $\mu_{(m,n)}$ – symmetric. Let U be a μ_m- open set and let $x \in U$. If $y \notin U$, then $x \notin c_{\mu_m}(\{y\})$, and so, by (X, μ_1, μ_2) is $\mu_{(m,n)}$ – symmetric, $y \notin c_{\mu_n}(\{x\})$. Hence $c_{\mu_n}(\{x\}) \subseteq U$. Thus (X, μ_1, μ_2) is $\mu_{(m,n)} - R_0$.

Theorem 3.1.26 Let (X, μ_1, μ_2) be a $\mu_{(1,2)} - R_0$ space and $\mu_{(2,1)} - R_0$ space. Then (X, μ_1, μ_2) is $\mu_{(m,n)} - T_1$ if and only if (X, μ_1, μ_2) is $\mu_{(m,n)} - T_0$.

Proof. $(\Rightarrow)$ Clearly.

$(\Leftarrow)$ Assume that (X, μ_1, μ_2) is $\mu_{(m,n)} - T_0$. Let x, y$\in$X with x $\neq$ y. Then there exists a subset U of X such that U is a μ_m- open set or a μ_n- open set, and x $\in$ U but y $\notin$ U or y $\in$U but x $\notin$U. Without loss of generality, we assume that U is μ_m- open and x $\in$U but y $\notin$ U. Then $\{y\} \cap U = \phi$ and so $x \notin c_{\mu_m}(\{y\})$. Since X is $\mu_{(1,2)} - R_0$ and $\mu_{(2,1)} - R_0$, $y \notin c_{\mu_n}(\{x\})$. Hence X- $\{c_{\mu_n}(\{x\})\}$ is a μ_n- open set containing y but not x. Hence (X, μ_1, μ_2) is $\mu_{(m,n)} - T_1$.

3.2 $\mu_{(m,n)} - T_2$ spaces

In this section, we introduce the notion of $\mu_{(m,n)} - T_2$ in bigeneralized topological spaces. Next, we study some properties of its.

Definition 3.2.1 A bigeneralized topological space (X, μ_1, μ_2) is called $\mu_{(m,n)} - T_2$ *or* $\mu_{(m,n)}$*- Hausdorff space* if for any $x, y \in X$ with $x \neq y$, there exist a μ_m- open set U and a μ_n- open set V such that $x \in U$, $y \in V$ and $U \cap V = \phi$.

Remark 3.2.2 It is clear that every $\mu_{(m,n)} - T_2$ space is a $\mu_{(m,n)} - T_1$ space. But a $\mu_{(m,n)} - T_1$ space is not a $\mu_{(m,n)} - T_2$ space, in general, as can be seen from the following example.

Example 3.2.3 Let $X = \{a, b, c\}$. We define two generalized topologies on X as follows: $\mu_1 = \{\phi, \{a, b\}, \{a, c\}, \{b, c\}, X\}$ and $\mu_2 = \{\phi, \{a\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$.
Then (X, μ_1, μ_2) is $\mu_{(1,2)} - T_1$ space but it is not $\mu_{(1,2)} - T_2$.

Example 3.2.4 Let $X = \{a, b, c, d\}$. We define two generalized topologies on X as follows: $\mu_1 = \{\phi, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}, X\}$ and $\mu_2 = P(X)$.
Then (X, μ_1, μ_2) is $\mu_{(1,2)} - T_2$.

Example 3.2.5 Let $X=\{a, b\}$. We define two generalized topologies on X as follows: $\mu_1=\{\phi,\{a\},X\}$ and $\mu_2=\{\phi,\{b\},X\}$.

Then (X, μ_1, μ_2) is not $\mu_{(1, 2)}$ - T_1 because if $b \neq a$ but there are no disjoint a μ_1 - open set containing b and a μ_2 - open set containing a.

Remark 3.2.6 Let (X, μ_1, μ_2) be a bigeneralized topological space. Then (X, μ_1, μ_2) is $\mu_{(m, n)}$ - T_2 if and only if for any $x, y \in X$ with $x \neq y$, there exist μ_m-open sets U_1, U_2 and μ_n-open sets V_1, V_2 such that

1. $x \in U_1$ and $y \in V_1$ and $U_1 \cap V_1 = \phi$,
2. $x \in V_2$ and $y \in U_2$ and $U_2 \cap V_2 = \phi$.

Theorem 3.2.7 For a bigeneralized topological space (X, μ_1, μ_2), the following are equivalent:

(1) X is a $\mu_{(m,n)}-T_2$ space.

(2) If $x \in X$, then for each $x \neq y$, then there exists a μ_m- open set U containing x such that $y \notin c_{\mu_n}(U)$.

(3) For each $x \in X$, $\{x\} = \cap\{c_{\mu_n}(U) : U \in \mu_m \text{ and } x \in U\}$.

Proof. $(1) \Rightarrow (2)$ Assume that X is a $\mu_{(m,n)}-T_2$ space and $x \in X$. Let y be a element of X such that $x \neq y$. Then there exist a μ_m- open set U and a μ_n- open set V such that $x \in U$, $y \in V$ and $U \cap V = \phi$. Thus $y \notin c_{\mu_n}(U)$.

$(2) \Rightarrow (3)$ Let $x \in X$. We will prove that $\{x\} = \cap\{c_{\mu_n}(U) : U \in \mu_m \text{ and } x \in U\}$. It is clear that $\{x\} \subseteq \cap\{c_{\mu_n}(U) : U \in \mu_m \text{ and } x \in U\}$. Let $y \in X$ with $y \neq x$. By assumption, there exists a μ_m - open set U_0 containing x such that $y \notin c_{\mu_n}(U_0)$. Then $\{y\} \notin \cap\{c_{\mu_n}(U) : U \in \mu_m \text{ and } x \in U\}$. Thus $\cap\{c_{\mu_n}(U) : U \in \mu_m \text{ and } x \in U\} \subseteq \{x\}$. Therefore, $\{x\} = \cap\{c_{\mu_n}(U) : U \in \mu_m \text{ and } x \in U\}$.

$(3)\Rightarrow(1)$ Assume that $\{x\}=\cap\{c_{\mu_n}(U):U\in\mu_m \text{ and } x\in U\}$ for each $x\in X$. Let $x,y\in X$ with $x\neq y$. Since $y\notin\{x\}=\cap\{c_{\mu_n}(U):U\in\mu_m \text{ and } x\in U\}$, there exists $U_0\in\mu_m$ such that $x\in U_0$ and $y\notin c_{\mu_n}(U_0)$. Since $y\notin c_{\mu_n}(U_0)$, there exists $V_0\in\mu_n$ such that $y\in V_0$ and $U_0\cap V_0=\phi$. Then $x\in U_0$, $y\in V_0$ and $U_0\cap V_0=\phi$. Hence, X is a $\mu_{(m,n)}-T_2$ space.

Theorem 3.2.8 Let (X,μ_1,μ_2) be a $\mu_{(m,n)}-R_1$ space. The following are equivalent:

(1) (X,μ_1,μ_2) is a $\mu_{(m,n)}-T_2$,

(2) (X,μ_1,μ_2) is a $\mu_{(m,n)}-T_1$,

(3) (X,μ_1,μ_2) is a $\mu_{(m,n)}-T_0$.

Proof. $(1)\Rightarrow(2)$ Clearly.

$(2)\Rightarrow(3)$ Obviously.

$(3)\Rightarrow(1)$ Assume that (X,μ_1,μ_2) is a $\mu_{(m,n)}-T_0$. To show that (X,μ_1,μ_2) is a $\mu_{(m,n)}-T_2$, let $x,y\in X$ with $x\neq y$. Since (X,μ_1,μ_2) is $\mu_{(m,n)}-R_1$ space, then (X,μ_1,μ_2) is pairwise R_0. By Theorem 3.1.17, we obtain that (X,μ_1,μ_2) is a $\mu_{(m,n)}-T_1$. Then $\{x\}$ is μ_m- closed and $\{y\}$ is μ_n- closed. Hence $c_{\mu_m}(\{x\})=\{x\}\neq\{y\}=c_{\mu_n}(\{y\})$. Since (X,μ_1,μ_2) is $\mu_{(1,2)}-R_1$ and $\mu_{(2,1)}-R_1$, there exist a μ_m- open set U and a μ_n- open set V such that $c_{\mu_m}(\{x\})\subseteq V$ and $c_{\mu_n}(\{y\})\subseteq U$ and $U\cap V=\phi$. Therefore, (X,μ_1,μ_2) is a $\mu_{(m,n)}-T_2$.

CHAPTER 4

$\mu_{(m,n)}$ - REGULAR SPACES AND $\mu_{(m,n)}$ - NORMAL SPACES

Throughout this chapter, we let $m, n \in \{1, 2\}$ with $m \neq n$.

In this chapter, we introduce the notion of $\mu_{(m,n)}$ - regular space and $\mu_{(m,n)}$ - normal space in bigeneralized topological spaces. Next, we study some properties of them.

4.1 $\mu_{(m,n)}$ - regular spaces

Definition 4.1.1 A bigeneralized topological space (X, μ_1, μ_2) is called a $\mu_{(m,n)}$- *regular space* if for any point $x \in X$ and for any μ_m - closed subset F of X with $x \notin F$, there exist $U \in \mu_m$ and $V \in \mu_n$ such that $x \in U$, $F \subseteq V$ and $U \cap V = \phi$.
A bigeneralized topological space (X, μ_1, μ_2) is said to be $\mu_{(m,n)} - T_3$ if (X, μ_1, μ_2) is $\mu_{(m,n)} - T_1$ and $\mu_{(m,n)}$ - regular.

Remark 4.1.2 The notion of $\mu_{(1,2)}$ - regular spaces [11] was introduced by Min.

Theorem 4.1.3 Every $\mu_{(m,n)} - T_3$ space is a $\mu_{(m,n)} - T_2$ space.

Proof. Let (X, μ_1, μ_2) be a $\mu_{(m,n)} - T_3$ space. Then (X, μ_1, μ_2) is a $\mu_{(m,n)} - T_1$ space and $\mu_{(m,n)}$ - regular space. We will prove that (X, μ_1, μ_2) is a $\mu_{(m,n)} - T_2$ space. Let x, y be elements of X such that $x \neq y$. Since (X, μ_1, μ_2) is a $\mu_{(m,n)} - T_1$ space, $\{y\}$ is a μ_m - closed set. Since (X, μ_1, μ_2) is a $\mu_{(m,n)}$ - regular space and $x \notin \{y\}$, there exist a μ_m - open set U and a μ_n - open set V such that $x \in U$, $y \in V$ and $U \cap V = \phi$.

Remark 4.1.4 In general, a $\mu_{(m,n)} - T_2$ space is not a $\mu_{(m,n)} - T_3$ space, as can be seen from the following example.

Example 4.1.5 Let $X = \{a, b, c, d\}$. We define two generalized topologies on X as follows:

$\mu_1 = \{\phi, \{c\}, \{d\}, \{a,b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a,b,c\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}, X\}$ and $\mu_2 = \{\phi, \{b\}, \{d\}, \{a,b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a,b,c\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}, X\}$. Then (X, μ_1, μ_2) is a $\mu_{(1, 2)} - T_2$ space but is not a $\mu_{(1, 2)} - T_3$ space.

Theorem 4.1.6 For a bigeneralized topological space (X, μ_1, μ_2), the following are equivalent:

(1) (X, μ_1, μ_2) is a $\mu_{(m, n)}$- regular space.

(2) For any point $x \in X$ and for any μ_m- closed set F with $x \notin F$, there are $U \in \mu_m$ and $V \in \mu_n$ such that $x \in U$, $F \subseteq V$ and $c_{\mu_n}(U) \cap V = \phi$.

(3) If $x \in X$ and F is μ_m- closed with $x \notin F$, then there is a μ_m- open set U containing x such that $c_{\mu_n}(U) \cap F = \phi$.

(4) If $x \in X$ and $G \in \mu_m$ with $x \in G$, then there is a μ_m- open set V containing x such that $x \in V \subseteq c_{\mu_n}(V) \subseteq G$.

(5) $F = \cap\{c_{\mu_m}(V) : V \in \mu_n \text{ and } F \subseteq V\}$ for each μ_m- closed subset F of X.

Proof. $(1) \Rightarrow (2)$ Let $x \in X$ and F be a μ_m- closed set such that $x \notin F$. Since X is $\mu_{(m, n)}$- regular space, there exist $U \in \mu_m$ and $V \in \mu_n$ such that $x \in U$, $F \subseteq V$ and $U \cap V = \phi$. Suppose that $c_{\mu_n}(U) \cap V \neq \phi$, say $y \in c_{\mu_n}(U) \cap V$. Then $y \in c_{\mu_n}(U)$ and $y \in V$. Since $V \in \mu_n$, $U \cap V \neq \phi$. which is a contradiction. Hence $c_{\mu_n}(U) \cap V = \phi$.

$(2) \Rightarrow (3)$ It is obvious.

$(3) \Rightarrow (4)$ Assume that $x \in X$ and $G \in \mu_m$ with $x \in G$. Then $X - G$ is a μ_m- closed set and $x \notin X - G$. By (3), there exists a μ_m- open set V containing x such that $c_{\mu_n}(V) \cap X - G = \phi$. Then $x \in V \subseteq c_{\mu_n}(V) \subseteq G$.

$(4) \Rightarrow (5)$ Let F be a μ_m- closed subset of X and let $y \notin F$. Then $y \in X - F$ and $X - F$ is μ_m- open. By (4), there is a μ_m- open set U containing y such that $y \in U \subseteq c_{\mu_n}(U) \subseteq X - F$. Then $F \subseteq X - c_{\mu_n}(U) \subseteq X - U$ and $y \notin X - U$. Set $W = X - c_{\mu_n}(U)$. Then W is μ_n- open and $F \subseteq W$. Since $X - U$ is μ_m- closed and $W \subseteq X - U$, $c_{\mu_m}(W) \subseteq X - U$. Thus $y \notin c_{\mu_m}(W)$. This implies $x \notin \cap\{c_{\mu_m}(V) : V \in \mu_n \text{ and } F \subseteq V\}$. Hence $\cap\{c_{\mu_m}(V) : V \in \mu_n \text{ and } F \subseteq V\} \subseteq F$. Therefore, $F = \cap\{c_{\mu_m}(V) : V \in \mu_n \text{ and } F \subseteq V\}$.

$(5)\Rightarrow(1)$ Let $x \in X$ and F be a μ_m - closed set such that $x \notin F$. By (5), there exists $V \in \mu_n$ such that $F \subseteq V$ and $x \notin c_{\mu_m}(V)$. Put $U = X - c_{\mu_m}(V)$. Then U is μ_m - open and $x \in U$. Moreover, $U \cap V = \phi$. Hence (X, μ_1, μ_2) is $\mu_{(m,n)}$ - regular.

Now, we recall $g\mu$ - closed sets and $g\mu$ - open sets in a generalized topological space. Let (X,μ) be a generalized topological space and A be a subset of X. Then A is called $g\mu$ - closed if $c_\mu(A) \subseteq U$ whenever $A \subseteq U$ and $U \in \mu$. And A is called $g\mu$ - open if $X-A$ is $g\mu$ - closed, or equivalently, $F \subseteq i_\mu(A)$ whenever $F \subseteq A$ and F is μ - closed.

Remark 4.1.7 Let (X,μ) be a generalized topological space and A be a subset of X. Then if A is μ - closed, then A is $g\mu$ - closed. And if A is μ - open, then A is $g\mu$ - open.

Example 4.1.8 Let $X = \{a,b,c\}$. We define generalized topologies on X as follow: $\mu = \{\phi, \{a\}, X\}$. Then $\{a,b\}$ is $g\mu$ - closed but it is not μ - closed. Similarly, $\{c\}$ is $g\mu$ - open but it is not μ - open.

Definition 4.1.9 Let (X,μ) be a generalized topological space and $A \subseteq X$. Then $c^*_\mu(A)$ denote the intersection of all $g\mu$ - closed sets containing A and $i^*_\mu(A)$ is the union of all $g\mu$ - open sets contained in A.

Lemma 4.1.10 Let (X,μ) be a generalized topological space and $A \subseteq X$. Then

(1) $x \in c^*_\mu(A)$ if and only if $V \cap A \neq \phi$ for each $g\mu$ - open V containing x.

(2) $x \in i^*_\mu(A)$ if and only if there exists a $g\mu$ - open V containing x such that $V \subseteq A$.

Proof. (1) $(\Rightarrow)$ Assume that there exists a $g\mu$ - open set V containing x such that $A \cap V = \phi$. Then $X-V$ is a $g\mu$ - closed set contained A and $x \notin X-V$. Hence $x \notin c^*_\mu(A)$.

$(\Leftarrow)$ Assume that $x \notin c^*_\mu(A)$. Then there exists a $g\mu$ - closed set V contained A such that $x \notin A$. Set $U = X-V$. Then U is $g\mu$ - open set containing x and $U \cap A = \phi$.

(2) $(\Rightarrow)$ Assume that $x \in i_{\mu}^{*}(A)$. Then there exists a $g\mu$ - open set V containing x such that $V \subseteq A$.

$(\Leftarrow)$ Assume that there exists a $g\mu$ - open set V containing x such that $V \subseteq A$. Then $x \in i_{\mu}^{*}(A)$.

Definition 4.1.11 Let (X, μ_1, μ_2) be a bigeneralized topological space. Then (X, μ_1, μ_2) is called a $g\mu_{(m,n)}$ *- regular space* if for any point $x \in X$ and for any μ_m - closed set F with $x \notin F$, there exist a $g\mu_m$ - open set U and a $g\mu_n$ - open set V such that $x \in U$, $F \subseteq V$ and $U \cap V = \phi$.

Remark 4.1.12 Every $\mu_{(m,n)}$ - regular space is a $g\mu_{(m,n)}$ - regular space. A $g\mu_{(m,n)}$ - regular space is not a $\mu_{(m,n)}$ - regular space, in general, as the following example.

Example 4.1.13 Let $X = \{a, b, c\}$. We define generalized topologies on X as follow: $\mu_1 = \{\phi, \{a, b\}, \{a, c\}, \{b, c\}, X\}$ and $\mu_2 = \{\phi, X\}$. Then (X, μ_1, μ_2) is a $g\mu_{(1,2)}$ - regular space but it is not $\mu_{(1,2)}$ - regular space.

Theorem 4.1.14 For a bigeneralized topological space (X, μ_1, μ_2), the following are equivalent:

(1) (X, μ_1, μ_2) is a $g\mu_{(m,n)}$ - regular space.

(2) For any point $x \in X$ and for any μ_m - closed set F with $x \notin F$, there exist a $g\mu_m$ - open set U and $g\mu_n$ - open set V such that $x \in U$, $F \subseteq V$ and $c_{\mu_n}^{*}(U) \cap V = \phi$.

Proof. $(1) \Rightarrow (2)$ Let $x \in X$ and F be a μ_m - closed set such that $x \notin F$. Since X is $g\mu_{(m,n)}$ - regular space, there exist a $g\mu_m$ - open set U and $g\mu_n$ - open set V such that $x \in U$, $F \subseteq V$ and $U \cap V = \phi$. Suppose that $c^{*}_{\mu_n}(U) \cap V \neq \phi$, say $y \in c^{*}_{\mu_n}(U) \cap V$. Then $y \in c^{*}_{\mu_n}(U)$ and $y \in V$. Since V is $g\mu_n$ - open $U \cap V \neq \phi$. which is a contradiction. Hence $c^{*}_{\mu_n}(U) \cap V = \phi$.

$(2) \Rightarrow (1)$ It is obvious.

Lemma 4.1.15 Let (X, μ_1, μ_2) be a bigeneralized topological space. The following are equivalent:

(1) If $x \in X$ and F is μ_m- closed with $x \notin F$, then there is a $g\mu_m$- open set U containing x such that $c^*_{\mu_n}(U) \cap F = \phi$.

(2) If $x \in X$ and $G \in \mu_m$ with $x \in G$, then there is a $g\mu_m$- open set V containing x such that $x \in V \subseteq c^*_{\mu_n}(V) \subseteq G$.

Proof. $(1) \Rightarrow (2)$ Assume that $x \in X$ and $G \in \mu_m$ with $x \in G$. Then $X - G$ is a μ_m- closed set and $x \notin X - G$. By (2), there exists a $g\mu_m$- open set V containing x such $c^*_{\mu_n}(V) \cap X - G = \phi$. Then $x \in V \subseteq c^*_{\mu_n}(V) \subseteq G$.

$(2) \Rightarrow (1)$ Assume that $x \in X$ and F is μ_m- closed with $x \notin F$. Then $X - F$ is μ_m- open and $x \in X - F$. By(3), there is a $g\mu_m$- open set V containing x such that $x \in V \subseteq c^*_{\mu_n}(V) \subseteq X - F$. Then $F \subseteq X - c^*_{\mu_n}(V)$, and so $c^*_{\mu_n}(V) \cap F = \phi$.

Proposition 4.1.16 Let (X, μ_1, μ_2) be a $g\mu_{(m, n)}$- regular space. Then

(1) If $x \in X$ and F is μ_m- closed with $x \notin F$, then there is a $g\mu_m$- open set U containing x such that $c^*_{\mu_n}(U) \cap F = \phi$.

(2) If $x \in X$ and $G \in \mu_m$ with $x \in G$, then there is a $g\mu_m$- open set V containing x such that $x \in V \subseteq c^*_{\mu_n}(V) \subseteq G$.

Proof. (1) Let $x \in X$ and F be a μ_m- closed set such that $x \notin F$. Since X is $g\mu_{(m, n)}$- regular space, there exist a $g\mu_m$- open set U and $g\mu_n$- open set V such that $x \in U$, $F \subseteq V$ and $U \cap V = \phi$. Suppose that $c^*_{\mu_n}(U) \cap F \neq \phi$, say $y \in c^*_{\mu_n}(U) \cap F$. Then $y \in c^*_{\mu_n}(U)$ and $y \in F$. Since $y \in F \subseteq V$ and V is $g\mu_n$- open, $U \cap V \neq \phi$ which is a contradiction. Hence $c^*_{\mu_n}(U) \cap F = \phi$.

(2) It is clear from lemma 4.1.15.

4.2 $\mu_{(m,n)}$ - normal spaces

Definition 4.2.1 Let (X, μ_1, μ_2) be a bigeneralized topological space. Then (X, μ_1, μ_2) is said to be $\mu_{(m,n)}$*- normal* if for any μ_m- closed set F and for any μ_n- closed set K with $F \cap K = \phi$, there exist $U \in \mu_m$ and $V \in \mu_n$ such that $F \subseteq V$, $K \subseteq U$ and $U \cap V = \phi$.
A space (X, μ_1, μ_2) is called $\mu_{(m,n)} - T_4$ if (X, μ_1, μ_2) is $\mu_{(m,n)} - T_1$ and $\mu_{(m,n)}$- normal.

Proposition 4.2.2 Let (X, μ_1, μ_2) be a bigeneralized topological space. If (X, μ_1, μ_2) is $\mu_{(m,n)} - T_4$, then (X, μ_1, μ_2) is $\mu_{(m,n)} - T_3$.

Proof. Assume that (X, μ_1, μ_2) is called $\mu_{(m,n)} - T_4$. We will prove that (X, μ_1, μ_2) is $\mu_{(m,n)} - T_3$. Let $x \in X$ and F a μ_m- closed such that $x \notin F$. Since (X, μ_1, μ_2) is $\mu_{(m,n)} - T_1$, $\{x\}$ is μ_n- closed. Since $\{x\} \cap F = \phi$ and (X, μ_1, μ_2) is $\mu_{(m,n)}$- normal, there exist a μ_m- open set U and a μ_n- open set V such that $\{x\} \subseteq U$, $F \subseteq V$ and $U \cap V = \phi$. Thus (X, μ_1, μ_2) is $\mu_{(m,n)} - T_3$.

Remark 4.2.3 A $\mu_{(m,n)} - T_3$ space is not $\mu_{(m,n)} - T_4$ space. as can be seen from the following example.

Example 4.2.4 In topology, let $\mathbb{R}$ be the set of all real numbers and B be the collection of all half-open intervals of the form

$$[a,b) = \{x \mid a \le x < b\},$$

Where $a < b$, the topology generated by B is call the lower limit topology on $\mathbb{R}$. When $\mathbb{R}$ is given the lower limite topology, we denote it by $\mathbb{R}_\ell$. Then the space $\mathbb{R}_\ell$ is regular, so the product space $\mathbb{R}_\ell^2$ is also regular. But $\mathbb{R}_\ell^2$ is not normal, see in [6].

Theorem 4.2.5 Let (X, μ_1, μ_2) be a bigeneralized topological space. The following are equivalent:

(1) (X, μ_1, μ_2) is $\mu_{(m,n)}$- normal space.

(2) If F is μ_m- closed and K is μ_n- closed such that $F \cap K = \phi$, then there are a μ_m- open set U and a μ_n- open set V such that $F \subseteq V$, $K \subseteq U$ and $c_{\mu_n}(U) \cap V = \phi$.

(3) If F is μ_m - closed and K is μ_n - closed such that $F \cap K = \phi$, then there exists a μ_m - open set U such that $K \subseteq U$ and $c_{\mu_n}(U) \cap F = \phi$.

(4) If F is μ_m - closed and G is μ_n - open such that $F \subseteq G$, then there is a μ_n - open set V such that $F \subseteq V \subseteq c_{\mu_m}(V) \subseteq G$.

Proof. $(1) \Rightarrow (2)$ Assume that F is a μ_m - closed set and K is a μ_n - closed set such that $F \cap K = \phi$. Since (X, μ_1, μ_2) is $\mu_{(m, n)}$ - normal space, there exist $U \in \mu_m$ and $V \in \mu_n$ such that $K \subseteq U$, $F \subseteq V$ and $U \cap V = \phi$. Suppose that $c_{\mu_n}(U) \cap V \neq \phi$, say $y \in c_{\mu_n}(U) \cap V$. Then $y \in c_{\mu_n}(U)$ and $y \in V$. Since $V \in \mu_n$, $U \cap V \neq \phi$. which is a contradiction. Hence $c_{\mu_n}(U) \cap V = \phi$.

$(2) \Rightarrow (3)$ It is obvious.

$(3) \Rightarrow (4)$ Assume that F is μ_m - closed and G is μ_n - open such that $F \subseteq G$. Then $X - G$ is a μ_n - open and $F \cap X - G = \phi$. By (3), there exists μ_m - open set U such that $X - G \subseteq U$ and $c_{\mu_n}(U) \cap F = \phi$. Hence $F \subseteq X - c_{\mu_n}(U) \subseteq X - U \subseteq G$. Let $V = X - c_{\mu_n}(U)$. Thus V is μ_n - open and $F \subseteq V \subseteq c_{\mu_m}(V) \subseteq X - U \subseteq G$.

$(4) \Rightarrow (1)$ Let F be a μ_m - closed set and K is a μ_n - closed set such that $F \cap K = \phi$. Then $X - K$ is a μ_n - open set and $F \subseteq X - K$. By (4), there exists a μ_n - open set V such that $F \subseteq V \subseteq c_{\mu_m}(V) \subseteq X - K$. Set $U = X - c_{\mu_m}(V)$. Then U is μ_m - open and $K \subseteq U$. Moreover, $U \cap V = X - c_{\mu_m}(V) \cap V \subseteq X - V \cap V = \phi$.
Then (X, μ_1, μ_2) is $\mu_{(m, n)}$ - normal.

CHAPTER 5

CONCLUSIONS AND RECOMMENDATIONS

5.1 CONCLUSIONS

This thesis has been concerned with some separation axioms in bigeneralized topological spaces. First, we constructed the concepts of the $\mu_{(m,n)}-T_0$ spaces, $\mu_{(m,n)}-T_1$ spaces, $\mu_{(m,n)}-R_0$ spaces, $\mu_{(m,n)}-R_1$ spaces, $\mu_{(m,n)}-T_2$ spaces, $\mu_{(m,n)}$- regular spaces, $\mu_{(m,n)}-T_3$ spaces, $g\mu_{(m,n)}$- regular spaces, $\mu_{(m,n)}$- normal spaces and $\mu_{(m,n)}-T_4$ spaces. Then we studied some properties of them. By definitions these spaces, we have the implications but reverse relation may be not true in general as follows:

$$\mu_{(m,n)}-T_4 \Rightarrow \mu_{(m,n)}-T_3 \Rightarrow \mu_{(m,n)}-T_2 \Rightarrow \mu_{(m,n)}-T_1 \Rightarrow \mu_{(m,n)}-T_0.$$

In particular, if (X,μ_1,μ_2) is $\mu_{(m,n)}-R_1$, then the following are equivalent:

$$\mu_{(m,n)}-T_2 \Leftrightarrow \mu_{(m,n)}-T_1 \Leftrightarrow \mu_{(m,n)}-T_0.$$

Further, We obtained some characterization of such spaces as follows:

(1) A bigeneralized topological space (X,μ_1,μ_2) is $\mu_{(m,n)}-T_0$ if and only if for each pair of distinct points x,y of X, $c_{\mu_m}(\{x\}) \neq c_{\mu_m}(\{y\})$ or $c_{\mu_n}(\{x\}) \neq c_{\mu_n}(\{y\})$.

(2) A bigeneralized topological space (X,μ_1,μ_2) is $\mu_{(m,n)}-T_1$ if and only if $\{x\}$ is μ_m- closed set and μ_n- closed set, for all $x \in X$.

(3) Let (X,μ_1,μ_2) be a bigeneralized topological space. If (X,μ_1,μ_2) is $\mu_{(m,n)}-R_1$, then (X,μ_1,μ_2) is $\mu_{(m,n)}-R_0$.

(4) The following are equivalent for a bigeneralized topological space (X,μ_1,μ_2).

(4.1) (X, μ_1, μ_2) is a $\mu_{(m,n)} - T_2$ space.

(4.2) If $x \in X$, then for each $x \neq y$, then exists a μ_m- open set U containing x such that $y \notin c_{\mu_n}(U)$.

(4.3) For each $x \in X$, $\{x\} = \cap\{c_{\mu_n}(U) : U \in \mu_m \text{ and } x \in U\}$.

(5) Every $\mu_{(m,n)} - T_3$ space is a $\mu_{(m,n)} - T_2$ space.

(6) The following are equivalent for a bigeneralized topological space (X, μ_1, μ_2).

(6.1) (X, μ_1, μ_2) is a $\mu_{(m,n)}$- regular space.

(6.2) For any point $x \in X$ and for any μ_m- closed set F with $x \notin F$, there are $U \in \mu_m$ and $V \in \mu_n$ such that $x \in U$, $F \subseteq V$ and $c_{\mu_n}(U) \cap V = \phi$.

(6.3) If $x \in X$ and F is μ_m- closed with $x \notin F$, then there is a μ_m- open set U containing x such that $c_{\mu_n}(U) \cap F = \phi$.

(6.4) If $x \in X$ and $G \in \mu_m$ with $x \in G$, then there is a μ_m- open set V containing x such that $x \in V \subseteq c_{\mu_n}(V) \subseteq G$.

(6.5) $F = \cap\{c_{\mu_m}(V) : V \in \mu_n \text{ and } F \subseteq V\}$ for each μ_m- closed subset F of X.

(7) Every $\mu_{(m,n)}$- regular space is a $g\mu_{(m,n)}$ - regular space. A $g\mu_{(m,n)}$- regular space is not a $\mu_{(m,n)}$- regular space.

(8) The following are equivalent for a bigeneralized topological (X, μ_1, μ_2) space.

(8.1) (X, μ_1, μ_2) is $\mu_{(m,n)}$- normal space.

(8.2) If F is μ_m- closed and K is μ_n- closed such that $F \cap K = \phi$, then there are a μ_m- open set U and a μ_n- open set V such that $F \subseteq V$, $K \subseteq U$ and $c_{\mu_n}(U) \cap V = \phi$.

(8.3) If F is μ_m- closed and K is μ_n- closed such that $F \cap K = \phi$, then there exists a μ_m- open set U such that $K \subseteq U$ and $c_{\mu_n}(U) \cap F = \phi$.

(8.4) If F is μ_m- closed and G is μ_n- open such that $F \subseteq G$, then there is a μ_n- open set V such that $F \subseteq V \subseteq c_{\mu_m}(V) \subseteq G$.

5.2 RECOMMENDATIONS

Although, we obtained several properties in the thesis, it still another interesting worth investigation further and we formulate the questions as follows.

(i) study new separation axioms in bigeneralized topological space.

(ii) study some separation axioms in biminimal structure space.

REFERENCES

REFERENCES

[1] Boonpok C. Weakly open functions on Bigeneralized topological spaces. Int. Journal of Math. Analysis 2010;4[18] : 891 – 897.

[2] Császár Á. Generalized open sets. Acta Math. Hungar. 1997;75: 65–87.

[3] Császár Á. Generalized topology, generalized continuity. Acta Math. Hungar. 2002;96: 351–357.

[4] Császár Á. Separation axioms for generalized topologies. Acta Math. Hungar. 2004;104: 63–69.

[5] Dungthaisong W., Boonpok C. and Viriyapong C. Int. Journal of Math. Analysis. 2011;5[24] :1175 –1184.

[6] J.R. Munkres. Topology. 2^{nd} edition. The United States of America; 2000.

[7] Min W. K. Some Results on generalized topological spaces and generalized systems. Acta Math. Hungar. 2005;108: 171–181.

[8] Min W. K. Almost continuity on generalized topological spaces. Acta Math. Hungar. 125(2009),121-125.

[9] Min W. K. Remark on separation axioms on generalized topological spaces. Journal of the chungcheong Mathematical society, 2010;23[2]: 293-298.

[10] Min W. K. A note on δ- and θ-modifications, Acta Math. Hungar. 2011; 132:107–112.

[11] Min W. K. Mixed weak continuity on on generalized topological spaces, Acta Math. Hungar. 2011;132:339–347.

[12] Roy B. On generalized R_0 and R_1 space. Acta Math. Hungar. 2010;127: 291–300.

[13] Sarsak M.S. New separation axioms in generalized topological spaces. Acta Math. Hungar. 2011;131: 120–121.

[14] Sarsak M.S. Weak separation axioms in generalized topological spaces. Acta Math. Hungar. 2011; 131[1-2]:110–121.

[15] Xun GE. and Ying GE. μ-Separations in generalized topological spaces. Math. J. Chinese Univ. 2010;25[2]: 243-252.

BIOGRAPHY

BIOGRAPHY

Name	Mrs. Pattharapohn Torton
Date of birth	May 26, 1981.
Place of birth	Mahasarsakham Province, Thailand
Institution Attended	Mattayom 6 in Sarakhampittayakom School in 1999, Mahasarsakham, Thailand.
	Bachelor of Science in Mathematics from Mahasarakham University in 2004, Thailand.
	Graduate Diploma Program in Teaching of Mathematics from Mahasarakham University in 2005, Thailand.
	Master of Science in Mathematics Education, Mahasarakham University in 2012, Thailand.
Contact address	House No 68, Village No.8, Nonghi sub-district, Wapipathum district, Mahasarakham Province, 44120.
	jong-d@hotmail.com