Connectedness in Ideal Generalized Topological Spaces

NATTHAYA BOONYAM

A Thesis Proposal submitted in partial fulfillment of the requirements for
the degree of Master of Science in Mathematics
at Mahasarakham University
June 2019

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The examining committee has unanimously approved this thesis, submitted by Miss Natthaya Boonyam, as a partial fulfillment of the requirements for the Master of Science in Mathematics at Mahasarakham University.

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ACKNOWLEDGEMENTS

This thesis would not have been accomplished if without the help from several people. First of all, I would like to thank Asst. Prof. Dr. Somnuek Worawiset, Asst. Prof. Chawalit Boonpok, Asst. Prof. Dr. Chokchai Viriyapong and Asst. Prof. Dr. Kittisak Saegsura for their kind comments and support as members of my dissertation committee.

I express my deepest sincere gratitude to my advisor, Asst. Prof. Dr. Daruni Boonchari and co-advisor, Asst. Prof. Jeeranunt Khampakdee, who introduced me to research. I am most grateful for their teaching and advice for their initial idea and encouragement which enable me to carry out my study research successfully. I would not have achieved this far and this thesis would not have been completed without all the support that I have always received from them.

I extend my thanks to all the lecturers who have taught me.

I would like to express my sincere gratitude to my parents for their understanding and help me for everything until this study completion.

Finally, I would like to thank all graduate students and staffs at the Department of Mathematics for supporting the preparation of this thesis.

Natthaya Boonyam

TITLE Connectedness in Ideal Generalized Topological Spaces

CANDIDATE Miss Natthaya Boonyam

DEGREE Master of Science **MAJOR** Mathematics

ADVISORS Asst. Prof. Daruni Boonchari, Ph.D.,
Asst. Prof. Jeeranunt Khampakdee, Ph.D.

UNIVERSITY Mahasarakham University **YEAR** 2019

ABSTRACT

In this research, we get the results of connectedness in set of ideal generalized topological spaces and a properties of ξ-I-μ-open in ideal generalized topological spaces. Moreover, we study the properties of μ-open set, μ-closed set, closure operator, interior operator, ξ-I-μ-open ξ-I-μ-closed in ideal generalized topological spaces and including characterization of ξ-I-μ-open, ξ-I-μ-closed set, ξ-I-μ-closure and ξ-I-μ-continuous.

Keywords : μ-open set, ξ-I-μ-open, ξ-I-μ-closed, ξ-I-μ-closure, ξ-I-μ-continuous.

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CHAPTER 1

INTRODUCTION

1.1 Background

In 1933, Kuratowski [9] and Vaidynathaswamy [14] introduced the concept of an ideal topological space. They also studied concept of localization theory. An ideal is a nonempty collection of subsets which closed under heredity and finite union.

In 2002, Császár [3] introduced the concept of generalized topological space (briefly GTS), that consisting of X and structure μ on X (briefly GT) such that μ is closed under arbitrary unions. Then (X, μ) is called a generalized topological space. He also introduced closure (c_μ) and interior (i_μ) in generalized topological spaces.

In 2008, Ekici and Noiri [5] introduced the concepts of connectedness in ideal topological spaces. He also studied the notions of separation axiom, connectedness and compactness.

In 2016, Modak [12] introduced the concepts of ideal generalized topological spaces. He also introduced the notions of the generalized closed sets in ideal generalized topological spaces. He obtained some properties of generalized closed sets in topological space, generalized topological space and ideal generalized topological space.

In 2018, Ekici [6] introduced the concept of a new type of open sets in ideal topological spaces called ξ-I-open sets by used the concepts of pre-I-open sets, semi-I-open sets and C_I^*-sets in ideal topological spaces.

For our purposes, we introduce the notion of connectedness in ideal generalized topological space. Moreover, we study some properties of connected sets, separated axioms, ξ-I-μ-open sets, ξ-I-μ-closed sets in ideal generalized topological spaces. We devide our work into 5 chapters, as follows:

In the first chapter, the introduction was presented.

In chapter 2, we present some basic concept and result of ideal generalized topological spaces without proof which are needed in the subsequent chapters.

In chapter 3, we introduce the concept of connectedness in ideal generalized topological space. We also study the basic properties of separated axioms and components in ideal generalized topological spaces.

In chapter 4, we introduced the concepts of ξ-I-μ-open sets and ξ-I-μ-closed sets in ideal generalized topological spaces. We also study the basic properties of strongly β-I-μ-open sets and $semi^*$-I-μ-open sets.

In the last chapter, we summarize result of our study.

CHAPTER 2

PRELIMINARIES

In this chapter, we will give some definitions, notations, dealing with some preliminaries and some useful results that will be duplicated in later chapter.

2.1 Ideal in topological spaces

This section discusses some properties of ideal in topological spaces.

Definition 2.1.1. [1] Let X be a nonempty set. A topology τ on X is a collection of subsets of X, each called an open set, such that

(1) $\emptyset$ and X are open sets.

(2) τ is closed under arbitrary unions, i.e. if $U_i \in \tau$ for $i \in I$ then $\bigcup\limits_{i\in I} U_i \in \tau$.

(3) τ is closed under finite intersection, i.e. if $U_1, U_2 \in \tau$ then $U_1 \cap U_2 \in \tau$.

The set X together with a topology τ on X is called a topological space and denoted by (X, τ).

Definition 2.1.2. [7] A nonempty collection I of subsets of a set X is said to be an ideal on X, if it satisfies the following two conditions:

(1) $A \in I$ and $B \subseteq A$ implies $B \in I$ (heredity).

(2) $A \in I$ and $B \in I$ implies $A \cup B \in I$ (finite additivity).

For the ideal I of X, the triple (X, τ, I) is called an ideal topological space.

Definition 2.1.3. [7] Let (X, τ, I) be an ideal topological space and $P(X)$ is the set of all subsets of X, a set operator $(\cdot)^* : P(X) \to P(X)$, called *a local function* with respect to τ and I, is defined as follows : for $A \subseteq X$

$$A^*(I, \tau) = \{x \in X : U \cap A \notin I, \forall U \in \tau(x)\}$$

where $\tau(x) = \{U \in \tau : x \in U\}$. We will simply write A^* for $A^*(\tau, I)$.

Theorem 2.1.4. [7] Let (X, τ, I) be an ideal topological spaces. If A and B are subsets of X and I, J are ideals on X. Then

(1) $A \subseteq B$ implies $A^* \subseteq B^*$.

(2) $J \subseteq I$ implies $A^*(\tau, I) \subseteq A^*(\tau, J)$.

(3) $A^* = cl(A^*) \subseteq cl(A)$ (A^* is closed subset of cl(A)).

(4) $(A^*)^* \subseteq A^*$.

(5) $(A \cup B)^* = A^* \cup B^*$.

(6) $A^* \setminus B^* = (A \setminus B)^* \setminus B^* \subseteq (A \setminus B)^*$.

(7) $U \in \tau$ implies $U \cap A^* = U \cap (U \cap A)^* \subseteq (U \cap A)^*$.

(8) $J \in \mathbb{I}$ implies $(A \cup J)^* = A^* = (A \setminus J)^*$

Definition 2.1.5. [5] Let (X, τ, I) be an ideal topological space. A Kuratowski closure operator Cl^* is defined by $Cl^*(A) = A \cup A^*$, for $A \subseteq X$. We will denote by $\tau^*(\tau, I)$ the topology generated by Cl^*, that is $\tau^*(\tau, I) = \{U \subseteq X : Cl^*(X \setminus U) = X \setminus U\}$. $\tau^*(\tau, I)$ is called $*$-topology structure which is finer than τ. We will simply write τ^* for $\tau^*(\tau, I)$.
The elements of $\tau^*(\tau, I)$ are called $*$-open and the complement of $*$-open sets are called $*$-closed.

Definition 2.1.6. [1] If (X, τ) is a topological space, then a subfamily $\mathbf{B} \subseteq \tau$ of the open sets is called a *base* (for the open sets) of the topology if

$$x \in U \subseteq X,\ U \text{ is open implies that there exists } B \in \mathbf{B} \text{ with } x \in B \subseteq U.$$

Theorem 2.1.7. [7] Let (X, τ, I) be an ideal topological space. The collection $\{V \setminus J : V \in \tau, J \in I\}$ is a basis for τ^*.

Theorem 2.1.8. [5] If (X, τ, I) is an ideal topological space and A is a subset of X, then (A, τ_A, I_A) is an ideal topological space, where τ_A is the relative topology on A and $I_A = \{A \cap J : J \in I\}$.

Lemma 2.1.9. [5] Let (X, τ, I) be an ideal topological space and $B \subseteq A \subseteq X$. Then $B^*(\tau_A, I_A) = B^*(\tau, I) \cap A$.

Lemma 2.1.10. [5] Let (X, τ, I) be an ideal topological space and $B \subseteq A \subseteq X$. Then $Cl^*_A(B) = Cl^*(B) \cap A$.

Definition 2.1.11. [5] A subset A of an ideal topological space (X, τ, I) is said to be $*$-dense if $Cl^*(A) = X$.

2.2 Connectedness in ideal topological spaces

This section discusses some properties of connectedness in ideal topological spaces.

Definition 2.2.1. [5] A topological space (X, τ) is said to be *connected* if X cannot be written as the disjoint union of two nonempty open sets.

Definition 2.2.2. [5] A topological space (X, τ) is said to be *hyperconnected* if every pair of nonempty open sets of X has nonempty intersection.

Definition 2.2.3. [5] An ideal topological space (X, τ, I) is call $*$-*connected* if X cannot be written as the disjoint union of a nonempty open set and a nonempty $*$-open set.

Definition 2.2.4. [5] An ideal topological space (X, τ, I) is called $*$-*hyperconnected* if A is $*$-dense for every nonempty subset A of X.

Lemma 2.2.5. [5] Let (X, τ, I) be an ideal topological space. For each $U \in \tau^*, (\tau^*)_U = (\tau_U)^*$.

Definition 2.2.6. [5] Let (X, τ, I) be an ideal topological space. $A, B \subseteq X$ are called $*$-*separated*if $Cl^*(A) \cap B = A \cap Cl(B) = \emptyset$.

Definition 2.2.7. [10] Let (X, τ, I) be an ideal topological space. $A, B \subseteq X$ are called $*$-*separated* if $A^* \cap Cl(B) = Cl(A) \cap B^* = A \cap B = \emptyset$.

Theorem 2.2.8. [5] Let (X, τ, I) be an ideal topological space. If A and B are $*$-*separated sets* of X and $A \cup B \in \tau$, then A and B are open and $*$-open, respectively.

Definition 2.2.9. [5] A subset A of an ideal topological space (X, τ, I) is called $*_s$-*connected* if A is not the union of two $*$-separated sets in (X, τ, I).

Theorem 2.2.10. [5] Let (X, τ, I) be an ideal topological space. If A is a $*_s$-connected set of X and H, G are $*$-separated sets of X with $A \subseteq H \cup G$, then either $A \subseteq H$ or $A \subseteq G$.

Theorem 2.2.11. [5] If A is a $*_s$-connected set of an ideal topological space (X, τ, I) and $A \subseteq B \subseteq Cl^*(A)$, then B is $*_s$-connected.

Corollary 2.2.12. [5] If A is a $*_s$-connected set of an ideal topological space (X, τ, I), then $Cl^*(A)$ is $*_s$-connected.

Definition 2.2.13. [5] Let (X, τ, I) be an ideal topological space and $x \in X$. The union of all $*_s$-connected subsets of X containing x is called the $*$-*component* of X containing x.

2.3 Ideal in generalized topological spaces

This section discusses some properties of ideal in generalized topological spaces.

Definition 2.3.1. [2] Let X be a nonempty set and $\mu \subseteq P(X)$. Then μ is called a generalized topology (in short, GT) on X if

(1) $\emptyset \in \mu$.

(2) $G_\alpha \in \mu$ for $\alpha \in M \neq \emptyset$ implies $G = \bigcup_{\alpha \in M} G_\alpha \in \mu$.

The pair (X, μ) is called *a generalized topological space* (in short, GTS) on X. The member of μ is called a μ-open set and the complement of a μ-open set is called a μ-closed set.

Definition 2.3.2. [3] Let (X, μ) be a generalized topological space and $A \subseteq X$. $c_\mu(A)$ is the intersection of all μ-closed sets containing A, and $i_\mu(A)$ is the union of all μ-open sets contained in A.

Definition 2.3.3. [12] Let (X, μ) be a generalized topological space. A mapping $()^{*\mu} : P(X) \to P(X)$ is defined as follows :

$$A^{*\mu}(\mu, I) = \{x \in X : U \cap A \notin I, \forall U \in \mu(x)\}$$

where $\mu(x) = \{U \in \mu : x \in U\}$.
The mapping is called the local function associated with the ideal I and generalized topology μ.
We will simply write $A^{*\mu}$ for $A^{*\mu}(\mu, I)$.
If I is an ideal on X, then (X, μ, I) is called *an ideal generalized topological space*.

Remark 2.3.4. [12] Let (X, μ, I) be an ideal generalized topological space and $A \subseteq X$. Then

(1) $A^{*\mu}(\mu, \{\emptyset\}) = c_\mu(A)$.

(2) $A^{*\mu}(\mu, P(X)) = \emptyset$.

(3) If $A \in I$, then $A^{*\mu} = \emptyset$.

(4) Neither $A \subseteq A^{*\mu}$ nor $A^{*\mu} \subseteq A$.

Theorem 2.3.5. [12] Let (X, μ, I) be an ideal generalized topological space. A, B are subsets of X and H, J are ideals on X. Then

(1) $(\emptyset)^{*\mu} = \emptyset$.

(2) If $A, B \subseteq X$ and $A \subseteq B$, $A^{*\mu} \subseteq B^{*\mu}$.

(3)$A^{*\mu} \subseteq c_\mu(A)$.

(4) $(A^{*\mu})^{*\mu} \subseteq c_\mu(A)$.

(5) $A^{*\mu}$ is a μ-closed set.

(6) $(A^{*\mu})^{*\mu} \subseteq A^{*\mu}$.

(7) $J \subseteq H$ implies $A^{*\mu}(H) \subseteq A^{*\mu}(J)$.

(8) $U \cap (U \cap A)^{*\mu} \subseteq U \cap A^{*\mu}$, for all $U \in \mu$.

(9) For $J \in I, (A \setminus J)^{*\mu} \subseteq A^{*\mu} = (A \cup J)^{*\mu}$.

Theorem 2.3.6. [4] Let (X, μ, I) be an ideal generalized topological space and A be a subset of X. Then

(1) If $A \in I$, then $A^{*\mu} = X \setminus M_\mu$ where M_μ is the union of all μ-open sets in generalized topological space (X, μ).

(2) If A is μ^*-closed, then $A^{*\mu} \subseteq A$.

Definition 2.3.7. [12] Let (X, μ, I) be an ideal generalized topological space. The set operator $c^{*\mu}$ is called a *generalized $*$-closure* and is defined as $c^{*\mu}(A) = A \cup A^{*\mu}$, for $A \subseteq X$. We will denoted by $\mu^*(\mu, I)$ the generalized structure, generated by $c^{*\mu}$ that is $\mu^*(\mu, I) = \{U \subseteq X : c^{*\mu}(X \setminus U) = X \setminus U\}$. $\mu^*(\mu, I)$ is called *$*$-generalized structure* which is finer than μ. We will simply write μ^* for $\mu^*(\mu, I)$.
The element of $\mu^*(\mu, I)$ are called μ^*-open and the complement of μ^*-open is called μ^*-closed.

Theorem 2.3.8. [12] The operator $c^{*\mu}$ satisfies following conditions:

(1) $A \subseteq c^{*\mu}(A)$ for $A \subseteq X$.

(2) $c^{*\mu}(\emptyset) = \emptyset$ and $c^{*\mu}(X) = X$.

(3) $c^{*\mu}(A) \subseteq c^{*\mu}(B)$ if $A \subseteq B \subseteq X$.

(4) $c^{*\mu}(A) \cup c^{*\mu}(B) \subseteq c^{*\mu}(A \cup B)$.

Remark. [12] If $I = \{\emptyset\}$, then $c^{*\mu}(A) = c_\mu(A)$ for $A \subseteq X$.

Definition 2.3.9. [12] A subset A of an ideal generalized topological space (X, μ, I) is said to be *μ^*-dense* in itself if $A \subseteq A^{*\mu}$.

Definition 2.3.10. [2] Let (X, μ) be a generalized topological space. Then X is *μ-connected* if X cannot be written as the disjoint union of two nonempty μ-open sets.

Example 2.3.11. Let $X = \{a, b, c\}$ and $\mu = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$.
Take $A = \{a\}$ and $B = \{b\}$.
Then $A \cap B = \emptyset$ and $A \cup B \neq X$.
Thus the space (X, μ) is μ-connected

Lemma 2.3.12. [6] For $A \subseteq X$ and A is a subset in ideal generalized topological spaces. Then

(1) $i^{*\mu}(A) = X \setminus c^{*\mu}(X \setminus A)$.

(2) $c^{*\mu}(A) = X \setminus i^{*\mu}(X \setminus A)$.

Definition 2.3.13. [6] Let S be a subset of a topological space (X, ϑ) with an ideal $\mathcal{L}$. S is said to be

(1) strongly β-I-open if $S \subseteq \hat{c}^*(\hat{i}(\hat{c}^*(S)))$.

(2) semi-I-open if $S \subseteq \hat{c}^*(\hat{i}(S))$.

(3) pre-I-open if $S \subseteq \hat{i}(\hat{c}^*(S))$.

(4) pre-I-closed if $X \setminus S$ is pre-I-open.

Definition 2.3.14. [8] A subset S of an ideal topological space (X, τ, I) is said to be β-I-open if $S \subseteq Cl(Int(Cl^*(S)))$. The complement of a β-I-open set is called β-I-closed.

Definition 2.3.15. [6] Let S be a subset of a topological space (X, ϑ) with an ideal $\mathcal{L}$. S is said to be

(1) semi*-I-open if $S \subseteq \hat{c}(\hat{i}^*(S))$.

(2) semi*-I-closed if $X \setminus S$ is semi*-I-open.

Definition 2.3.16. [8] A function f : $(X, \tau, I) \to (Y, \sigma)$ is said to be strongly β-I-continuous if for each $x \in X$ and each open set V of Y containing $f(x)$, there exists $U \in \beta IO(X, x)$ such that $f({}_{\beta I}Cl(U) \subseteq Cl(V)$.

CHAPTER 3

CONNECTEDNESS IN IDEAL GENERALIZED TOPOLOGICAL SPACES

In this section, we study some properties of connectedness in ideal generalized topological spaces.

3.1 Connectedness in ideal generalized topological spaces

In this section, we get results of connected in ideal generalized topological spaces.

Throughout this thesis μ will represent a generalized topological spaces such that $\emptyset, X \in \mu$ and the union of elements of μ belong to μ.

Theorem 3.1.1. Let (X, μ, I) be an ideal generalized topological space. Then the collection set $\{M \setminus H : M \in \mu, H \in I\}$ is a basis for $\mu^*(\mu, I)$.

Proof. Let (X, μ, I) be an ideal generalized topological space and $x \in G \in \mu^*$.
By Definition 2.3.6, we have $(X \setminus G)^* \subseteq X \setminus G$.
Since $x \in G$, we have $x \notin X \setminus G$, this implies that $x \notin (X \setminus G)^*$.
So, there exists $O \in \mu(x)$ such that $(X \setminus G) \cap O \in I$.
$x \in O \setminus ((X \setminus G) \cap O) = O \cap G \subseteq G$.
It follows that $O \setminus ((X \setminus G) \cap O) \in \{M \setminus H : M \in \mu$ and $H \in I\}$, and $x \in O \setminus ((X \setminus G) \cap O) \subseteq G$.
We get that $\{M \setminus H : M \in \mu$ and $H \in I\}$ is a basis for $\mu^*(\mu, I)$. □

Example 3.1.2. Let $X = \{a, b, c, d\}$ and $\mu = \{\emptyset, X, \{a, b\}, \{a, c\}, \{a, b, c\}\}$, $I = \{\emptyset, \{a\}, \{d\}, \{a, d\}\}$.
Then $\{\emptyset, X, \{b\}, \{c\}, \{b, c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{b, c, d\}\}$ is a basis for $\mu^*(\mu, I)$.

Theorem 3.1.3. Let (X, μ, I) be an ideal generalized topological space and M is a subset of X, then (M, μ_M, I_M), where μ_M is the relative generalized topology on M and $I_M = \{M \cap H : H \in I\}$ is an ideal space.

Proof. Let $M \subseteq X$ and μ_M be a relative generalized topology on M.

Since $\emptyset \in I$, then $\emptyset \cap M \in I_M$. So $\emptyset \in I_M$.

Let $K \in I_M$ and $N \subseteq K$, then $K = M \cap H$ for some $H \in I$.

Since $N \subseteq K$, then $N \subseteq M \cap H$ and $N = N \cap (M \cap H) = M \cap (N \cap H)$.

Since $N \cap H \subseteq H$ and $H \in I$, we get that $N \cap H \in I$.

This implies that $N \in I_M$.

Since $K \in I_M$ and $N \in I_M$, then $K = M \cap H_1$ and $N = M \cap H_2$ for some $H_1, H_2 \in I$.

So $K \cup N = (M \cap H_1) \cup (M \cap H_2)$

$= M \cap (H_1 \cup H_2) \in I_M$.

Since $H_1 \cup H_2 \in I$, we have $K \cup N \in I_M$.

Therefore I_M is an ideal.

Hence (M, μ_M, I_M) is an ideal generalized topological space. $\square$

Lemma 3.1.4. Let (X, μ, I) be an ideal generalized topological space and $N \subseteq M \subseteq X$. Then $N^{*\mu}(\mu_M, I_M) = N^{*\mu}(\mu, I) \cap M$.

Proof. $(\Rightarrow)$ Let $x \in N^{*\mu}(\mu_M, I_M)$.

Suppose $x \notin N^{*\mu}(\mu, I) \cap M$. This implies that $x \notin N^{*\mu}(\mu, I)$ or $x \notin M$

Since $x \notin N^{*\mu}(\mu, I)$, then there exists μ-open set U containing x such that $U \cap N \in I$.

As U is μ-open and $M \subseteq X$, then $U \cap M \in \mu_M$, and $(U \cap M) \cap N = (U \cap N) \cap M$.

This implies that $(U \cap N) \cap M \in I_M$.

So $(U \cap M) \cap N \in I_M$.

Hence $x \notin N^{*\mu}(\mu_M, I_M)$. This is a contradiction.

Therefore $N^{*\mu}(\mu_M, I_M) \subseteq N^{*\mu}(\mu, I)$

$(\Leftarrow)$ Let $x \in N^{*\mu}(\mu, I) \cap M$.

Assume that $x \notin N^{*\mu}(\mu_M, I_M)$.

Then there exists $K \in \mu_M(x)$ such that $K \cap N \in I_M$.

And we have that there exists $G \in I$ and $M \subseteq X$.

So $G \cap M \in I_M$.

Since $G \cap M \subseteq G$ and $I_M \subseteq I$, we get that $K \cap N \in I$.

Therefore $x \notin N^{*\mu}(\mu, I)$. This is a contradiction.

Hence $N^{*\mu}(\mu, I) \cap M \subseteq N^{*\mu}(\mu_M, I_M)$. $\square$

Lemma 3.1.5. Let (X, μ, I) be an ideal generalized topological space and $B \subseteq A \subseteq X$. Then $C_A^{*\mu}(B) = C^{*\mu}(B) \cap A$.

Proof. $(\Rightarrow)$ Let $x \in C_A^{*\mu}(B)$.

Since $C_A^{*\mu}(B) = B^{*\mu}(\mu_A, I_A) \cup B$, then $x \in B^{*\mu}(\mu_A, I_A) \cup B$ i.e. $x \in B^{*\mu}(\mu_A, I_A)$ or $x \in B$.

Since $x \in B^{*\mu}(\mu_A, I_A)$, then by Lemma 3.1.3, we get that $B^{*\mu}(\mu_A, I_A) = B^{*\mu}(\mu, I) \cap A$.

So $x \in B^{*\mu}(\mu, I) \cap A$.

$$\begin{aligned} \text{Thus } B^{*\mu}(\mu_A, I_A) \cup B &= (B^{*\mu}(\mu, I) \cap A) \cup B \\ &= (B^{*\mu}(\mu, I) \cup B) \cap (A \cup B) \\ &= C^{*\mu}(B) \cap A \end{aligned}$$

Hence $x \in c^{*\mu}(B) \cap A$

$(\Leftarrow)$ Let $y \in c^{*\mu}(B) \cap A$ i.e, $y \in c^{*\mu}(B)$ and $y \in A$.

Since $y \in c^{*\mu}(B)$ and $c^{*\mu}(B) = B^{*\mu}(\mu, I) \cup B$, we have that $y \in B^{*\mu}(\mu, I)$ or $y \in B$

Suppose that $y \notin B^{*\mu}(\mu, I)$.

Then there exists a μ-open set U containing y such that $U \cap B \in I$.

Since $y \in A$, then $(U \cap B) \cap A \in I_A$.

So $(U \cap B) \cap A = (U \cap A) \cap B \in I_A$, for some $U \cap A \in \mu_A(y)$.

Thus $y \notin B^{*\mu}(\mu_A, I_A)$. This is a contradiction.

Therefore $y \in B^{*\mu}(\mu_A, I_A)$ or $y \in B$.

Hence $y \in B^{*\mu}(\mu_A, I_A) \cup B$ i.e. $y \in c_A^{*\mu}(B)$. □

Definition 3.1.6. Let (X, μ) be a generalized topological space. Then X is *μ-hyperconnected* if every pair of nonempty μ-open sets of X has nonempty intersection.

Example 3.1.7. Let $X = \{a, b, c, d\}$ and $\mu = \{\emptyset, \{a, b\}, \{a, c\}, \{a, b, c\}\}$, $I = \{\emptyset, \{a\}, \{d\}, \{a, d\}\}$.

$A = \{a, b\}$ and $B = \{a, c\}$. Then $A \cap B = \{a\} \neq \emptyset$.

$A = \{a, b\}$ and $B = \{a, b, c\}$. Then $A \cap B = \{a, b\} \neq \emptyset$.

$A = \{a, c\}$ and $B = \{a, b, c\}$. Then $A \cap B = \{a, c\} \neq \emptyset$.

Therefore X is μ-hyperconnected.

Definition 3.1.8. An ideal generalized topological space (X, μ, I) is called μ^*-*connected* if X cannot be written as the disjoint union of a nonempty μ-open set and a nonempty μ^*-open set.

Definition 3.1.9. An ideal generalized topological space (X, μ, I) is said to be μ^*-*hyperconnected* if A is μ^*-dense for every nonempty μ-open set A of X.

Example 3.1.10. Let $X = \{a, b, c, d\}$ and $\mu = \{\emptyset, X, \{a, b\}, \{b, c\}, \{a, b, c\}\}$,
$I = \{\emptyset, \{b\}\}$. $\mu^* = \{\emptyset, X, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}, \{a, c, d\}\}$.
Take $A = \{a, c\}$ and $B = \{a, b\}$.
Then $A \cup B = \{a, b, c\} \neq X$. Therefore (X, μ, I) is μ^*-connected.

Example 3.1.11. Let $X = \{a, b, c, d\}$ and $\mu = \{\emptyset, X, \{a, b\}, \{a, c\}, \{a, b, c\}\}$, $I = \{\emptyset, \{a\}, \{d\}, \{a, d\}\}$.
Take $A = \{a, b\}$ and $B = \{a, c\}$.
Then $A \cap B = \{a\} \neq \emptyset$.
Therefore (X, μ) is μ-hyperconnected.

Remark 3.1.12. The following implications hold for an ideal space (X, μ, I).

$$\begin{array}{ccc}
(X,\mu,I)\ is\ \mu^* - hyperconnected & \Longrightarrow & (X,\mu)\ is\ \mu - hyperconnected \\
\Downarrow & & \Downarrow \\
(X,\mu,I)\ is\ \mu^* - connected & \Longrightarrow & (X,\mu)\ is\ \mu - connected
\end{array}$$

Above implications are not reversible as can be seen from the following example.

Example 3.1.13. Let $X = \{a, b, c\}$ and $\mu = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$.
Take $A = \{a\}$ and $B = \{b\}$, then $A \cap B = \emptyset$ and $A \cup B \neq X$.
Thus the space (X, μ) is μ-connected but (X, μ) is not μ-hyperconnected.

Example 3.1.14. Let $X = \{a, b, c, d\}$ and $\mu = \{\emptyset, X, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$,
$I = \{\emptyset, \{b\}\}$.
Take $A = \{c\}$ and $B = \{a, b\}$ such that $A \cap B = \emptyset$, then $A \cup B = \{a, b, c\} \neq X$.
But since $A = \{c\}$ and $A^{*\mu} = \{c, d\}$, then $c^{*\mu}(A) = \{c, d\} \neq X$.
Then the space (X, μ, I) is μ^*-connected but (X, μ, I) is not μ^*-hyperconnected.

Example 3.1.15. Let $X = \{a,b,c,d\}$ and $\mu = \{\emptyset, \{a,b\}, \{a,c\}, \{a,b,c\}\}$, $I = \{\emptyset, \{a\}, \{d\}, \{a,d\}\}$.

Take $A = \{a,c\}$ and $B = \{a,b,c\}$, then $A \cap B = \{a,c\}$.

But if $A = \{a,b\}$, then $A^{*\mu} = \{b\}$. We get that $c^{*\mu}(A) = \{a,b\} \neq X$.

$B = \{a,c\}$, then $B^{*\mu} = \{c\}$. We get that $c^{*\mu}(B) = \{a,c\} \neq X$.

$C = \{a,b,c\}$, then $C^{*\mu} = \{a,b,c\}$. We get that $c^{*\mu}(C) = \{a,b,c\} \neq X$.

Then the space (X,μ) is μ-hyperconnected but (X,μ,I) is not μ^*-hyperconnected.

Example 3.1.16. Let $X = \{a,b,c,d\}$ and $\mu = \{\emptyset, \{c\}, \{a,b\}, \{b,c\}, \{a,b,c\}\}$, $I = \{\emptyset, \{b\}\}$.

Take $A = \{a,b\}$ and $B = \{c\}$ such that $A \cap B = \emptyset$, then $A \cup B \neq X$.

But since $G = \{a,b,c\}$ and $H = \{d\}$ such that $G \cap H = \emptyset$, then $G \cup H = X$

Then the space (X,μ) is μ-connected but (X,μ,I) is not μ^*-connected.

Lemma 3.1.17. Let (X,μ,I) be an ideal generalized topological space. For each $M \in \mu^*, (\mu^*)_M \subsetneq (\mu_M)^*$.

Proof. Let $P \in (\mu^*)_M$ and $M \in \mu^*$.

Then there exists $K \in \mu^*$ such that $K \cap M = P$.

Since $M \in \mu^*$, we have $c^{*\mu}(X \setminus M) = X \setminus M$ i.e, $(X \setminus M)^* \subseteq X \setminus M$ and $M \subseteq X \setminus (X \setminus M)^*$.

And $K \in \mu^*$, we have $c^{*\mu}(X \setminus K) = X \setminus K$ i.e, $(X \setminus K)^* \subseteq X \setminus K$ and $K \subseteq X \setminus (X \setminus K)^*$.

$$
\begin{aligned}
\text{So, } P = K \cap M &\subseteq (X \setminus (X \setminus K)^*) \cap (X \setminus (X \setminus M)^*) \\
&= X \setminus ((X \setminus K)^* \cup (X \setminus M)^*) \\
&\subseteq X \setminus ((X \setminus K) \cup (X \setminus M))^* \\
&= X \setminus (X \setminus (K \cap M))^* \\
&= X \setminus (X \setminus P)^*.
\end{aligned}
$$

Thus $(X \setminus P)^* \subseteq X \setminus P$.

This implies that $P \in \mu^*$.

$$
\begin{aligned}
\text{Hence } (X \setminus P)^* \cap M &\subseteq (X \setminus P) \cap M \\
&= M \setminus P.
\end{aligned}
$$

As $P \subseteq M \setminus ((X \setminus P)^* \cap M) = M \setminus (X \setminus P)^*$.

Therefore $(M \setminus P)^* \subseteq (X \setminus P)^* \subseteq M \setminus P$.

So, we get that $c^{*\mu}(M \setminus P) = M \setminus P$ i.e, $P \in (\mu_M)^*$. $\square$

Definition 3.1.18. Let (X, μ, I) be an ideal generalized topological space. M and K are called *μ^*-separated* if $c^{*\mu}(M) \cap K = M \cap c_\mu(K) = \emptyset$.

Definition 3.1.19. Let (X, μ) be a generalized topological space and (Y, μ_Y) be a subspace of X, $M \subseteq Y \subseteq X$. Then $c_{\mu_Y}(K)$ is the intersection of all μ_Y-closed sets containing A.

Definition 3.1.20. Let (X, μ, I) be an ideal generalized topological space and (Y, μ_Y, I_Y) be a subspace of X. Nonempty subsets M, K of an ideal generalized topological space (Y, μ_Y, I_Y) are called μ^*-separated in Y if $c_Y^{*\mu}(M) \cap K = M \cap c_{\mu_Y}(K) = \emptyset$.

Theorem 3.1.21. Let (X, μ, I) be an ideal generalized topological space. If M and K are μ^*-separated sets of X and $M \cup K \in \mu$, then M and K are μ-open and μ^*-open, respectively.

Proof. Let M and K are μ^*-separated in X and $M \cup K \in \mu$.
Then $c^{*\mu}(M) \cap K = M \cap c_\mu(K) = \emptyset$.
If $c^{*\mu}(M) \cap K = \emptyset$, then $K \subseteq X \setminus c^{*\mu}(M)$.
Hence $(M \cup K) \cap K \subseteq (M \cup K) \cap (X \setminus c^{*\mu}(M))$.
Since
$$\begin{aligned}(M \cup K) \cap (X \setminus c^{*\mu}(M)) &= (M \cup K) \cap (X \setminus (M \cup M^{*\mu})).\\
&= (M \cup K) \cap ((X \setminus M) \cap (X \setminus M^{*\mu})).\\
&= ((M \cup K) \cap (X \setminus M)) \cap (X \setminus M^{*\mu})\\
&= ((M \cup K) \cap M^c) \cap (X \setminus M^{*\mu})\\
&= ((M \cap M^c) \cup (K \cap M^c)) \cap (X \setminus M^{*\mu})\\
&= (\emptyset \cup (K \cap M^c)) \cap (X \setminus M^{*\mu})\\
&= (K \cap M^c) \cap (M^{*\mu})^c\\
&= K \cap (M^c \cap (M^{*\mu})^c)\\
&= K \cap (M \cup M^{*\mu})^c\\
&= K \cap (X \setminus c^{*\mu}(M))\end{aligned}$$
and $K \subseteq X \setminus c^{*\mu}(M)$, it follows that $(M \cup K) \cap (X \setminus c^{*\mu}(M)) = K \cap (X \setminus c^{*\mu}(M)) = K$
.
As $M \cup K \in \mu$ such that $M \cup K$ is μ-open and $\mu \subseteq \mu^*$, we have $M \cup K \in \mu^*$.
And M and K are μ^*-separated in X.
So $\emptyset = c^{*\mu}(M) \cap K = (M \cup M^{*\mu}) \cap K = (M \cap K) \cup (M^{*\mu} \cap K)$.

We get that $M \cap K = \emptyset$ or $M^{*\mu} \cap K = \emptyset$.

If $M \cap K = \emptyset$ implies that $M \subseteq X \setminus K$, then $c^{*\mu}(M) \subseteq c^{*\mu}(X \setminus K)$ and $c^{*\mu}(X \setminus K) = X \setminus K$ i.e. K is μ^*-open.

As $M \cap c_\mu(K) = \emptyset$, we get that $M \subseteq X \setminus c_\mu(K)$.

Thus $c_\mu(K)$ is μ-closed by definition, $X \setminus c_\mu(K)$ is μ-open set.

Hence M is μ-open set. We get that K is μ^*-open likewise. □

Theorem 3.1.22. Let (X, μ, I) be an ideal generalized topological space and $M, K \subseteq Y \subseteq X$. Then M, K are μ^*-separated in Y if and only if M, K are μ^*-separated in X.

Proof. $(\Rightarrow)$ Suppose that M, K are μ^*-separated in Y.

By Lemma 3.1.4, we get that $c_Y^{*\mu}(M) \cap K = (c^{*\mu}(M) \cap Y) \cap K = c^{*\mu}(M) \cap (Y \cap K)$.

since $K \subseteq Y$, then $c^{*\mu}(M) \cap (Y \cap K) = c^{*\mu}(M) \cap K$.

Thus $c^{*\mu}(M) \cap K = c_Y^{*\mu}(M) \cap K = \emptyset = M \cap c_Y^{\mu}(K) = M \cap (c_\mu(K) \cap Y) = (M \cap Y) \cap c_\mu(K) = M \cap c_\mu(K)$.

Therefore M, K are μ^*-separated in X.

$(\Leftarrow)$ Suppose that subsets M, K are μ^*-separated in X.

Since $M, K \subseteq Y$ and by Definition 3.1.12, we get that $c^{*\mu}(M) \cap K = M \cap c_\mu(K) = \emptyset$.

So
$$
\begin{aligned}
c_Y^{*\mu}(M) \cap K &= (c^{*\mu}(M) \cap Y) \cap K \\
&= c^{*\mu}(M) \cap (Y \cap K) \\
&= c^{*\mu}(M) \cap K \\
&= \emptyset \\
&= M \cap c_\mu(K) \\
&= (M \cap Y) \cap c_\mu(K) \\
&= M \cap (Y \cap c_\mu(K)) \\
&= M \cap c_Y^{\mu}(K)
\end{aligned}
$$

Thus $c_Y^{*\mu}(M) \cap K = \emptyset = M \cap c_Y^{\mu}(K)$.

Therefore M, K are μ^*-separated in Y. □

Definition 3.1.23. A subset M of an ideal generalized topological spaces (X, μ, I) is called μ^{*s}-connected if M is not the union of two μ^*-separated sets in (X, μ, I).

Example 3.1.24. Let $X = \{a, b, c, d\}, \mu = \{\emptyset, X, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$. And $I = \{\emptyset, \{b\}\}$.

μ-closed set $= \{\emptyset, X, \{b\}, \{d\}, \{b,d\}, \{c,d\}, \{a,b,d\}, \{b,c,d\}\}$. Let $M = \{b\}$ and $M^{*\mu} = \{b\}$, then $c^{*\mu}(M) = \{b\}$.

And $K = \{c,d\}$, then $c_\mu(K) = \{c,d\}$.

So $c^{*\mu}(M) \cap K = \emptyset = M \cap c_\mu(K)$.

Thus M and K are μ^*-separated.

Therefore $M \cup K = \{b,c,d\}$ is not μ^{*s}-connected set.

Theorem 3.1.25. Let (X, μ, I) be an ideal generalized topological space. If M is a μ^{*s}-connected set in X and H, K are μ^*-separated sets in X with $M \subseteq H \cup K$, then either $M \subseteq H$ or $M \subseteq K$.

Proof. Let M be a μ^{*s}-connected in X and H, K are μ^*-separated sets in X with $M \subseteq H \cup K$.

Then $M = M \cap (H \cup K) = (M \cap H) \cup (M \cap K)$.

Since H, K are μ^*-separated, we have $c^{*\mu}(H) \cap K = H \cap c_\mu(K) = \emptyset$.

Thus $(M \cap K) \cap c^{*\mu}(M \cap H) \subseteq K \cap c^{*\mu}(H) = \emptyset$.

Likewise, we have $(M \cap H) \cap c_\mu(M \cap K) = \emptyset$.

If $M \cap H \neq \emptyset$ and $M \cap K \neq \emptyset$, then $M \cap H$ and $M \cap K$ are μ^*-separated sets in X.

So, $M = (M \cap H) \cup (M \cap K)$.

Thus M is not μ^{*s}-connected.

This is a contradiction.

Hence either $M \cap H$ or $M \cap K$ are empty.

Assume that $M \cap H = \emptyset$.

Then $M = M \cap K$ implies that $M \subseteq K$.

In a similar way, we get that $M \subseteq H$. $\square$

Theorem 3.1.26. If M is a μ^{*s}-connected set of an ideal generalized topological space (X, μ, I) and $M \subseteq N \subseteq c^{*\mu}(M)$, then N is μ^{*s}-connected.

Proof. Assume that N is not μ^{*s}-connected.

Then there exists μ^*-separated sets A and B such that $N = A \cup B$ i.e, A, B are nonempty and $c^{*\mu}(A) \cap B = A \cap c_\mu(B) = \emptyset$.

By Theorem 3.1.18, either $N \subseteq A$ or $N \subseteq B$.

Assume that $N \subseteq A$.

Then $c^{*\mu}(N) \subseteq c^{*\mu}(A)$ and $B = B \cap c^{*\mu}(N) \subseteq B \cap c^{*\mu}(A) = \emptyset$.

Thus B is empty set. This is a contradiction.

Suppose $N \subseteq B$.

Then $c_\mu(N) \subseteq c_\mu(B)$.

So $A = A \cap c_\mu(N) \subseteq A \cap c_\mu(B) = \emptyset$.

Thus A is empty set. This is a contradiction.

Therefore N is μ^{*s}-connected. $\square$

Corollary 3.1.27. If M is a μ^{*s}-connected set of an ideal generalized topological space (X, μ, I), then $c^{*\mu}(M)$ is μ^{*s}-connected.

Theorem 3.1.28. Let (X, μ, I) be an ideal generalized topological space. If $\{M_n : N \in \Lambda\}$ is a nonempty family of μ^{*s}-connected sets with $\bigcap_{n\in\Lambda} M_n \neq \emptyset$, then $\bigcup_{n\in\Lambda} M_n$ is a μ^{*s}-connected set.

Proof. Suppose that $\bigcup_{n\in\Lambda} M_n$ is not μ^{*s}-connected.

Then we have that $\bigcup_{n\in\Lambda} M_n = A \cup B$ where A and B are μ^*-separated sets.

Since $\bigcap_{n\in\Lambda} M_n \neq \emptyset$, we have a point $x \in \bigcap_{n\in\Lambda} M_n$.

Thus $x \in M_n$ and $M_n \subseteq A \cup B$ for all $n \in \Lambda$.

It follows from $A \cap B = \emptyset$, that either $x \in A$ or $x \in B$.

In case $x \in A$; For any $n \in \Lambda$, $M_n \cap A \neq \emptyset$.

By Theorem 3.1.18, $M_n \subseteq A$ or $M_n \subseteq B$.

So $M_n \subseteq A$ for all $n \in \Lambda$, and then $\bigcup_{n\in\Lambda} M_n \subseteq A$.

This implies that $B = \emptyset$.

This is a contradiction.

In case $x \in B$; in the same way, we have $A = \emptyset$. This is a contradiction.

Hence $\bigcup_{n\in\Lambda} M_n$ is μ^{*s}-connected. $\square$

Definition 3.1.29. Let (X, μ, I) be an ideal generalized topological space and $x \in X$. The union of all μ^{*s}-connected subsets of X containing x is called the μ^*-component of X containing x.

Theorem 3.1.30. Every μ^*-component of an ideal generalized topological space (X, μ, I) is a maximal μ^{*s}-connected set.

Proof. Let $x \in X$.

Suppose that C_x is μ^*-component of X such that $x \in C_x$.

So $C_x = \bigcup_{j\in J}\{M_j \subseteq X : M_j$is a μ^{*s}- connected set containing x$\}$.

Since M_j is a μ^*-connected for all $j \in J$, by Theorem 3.1.21, we get that $\bigcap_{j\in J}\{M_j \subseteq X : x \in M_j\} \neq \emptyset$.

Hence C_x is μ^{*s}-connected.

Next, let $A \subseteq X$ and A is μ^{*s}-connected such that $C_x \subseteq A$.

Then $x \in A$, by definition of C_x we have $A \subseteq C_x$.

Thus $C_x = A$.

Therefore C_x is a maximal μ^{*s}-connected set of X. □

Theorem 3.1.31. The set of all distinct μ^*-component of an ideal generalized topological space (X, μ, I) forms a partition of X.

Proof. Let M and K be two distinct μ^*-components of X.

Suppose that $M \cap K \neq \emptyset$.

By Theorem 3.1.20, $M \cup K$ is μ^{*s}-connected in X.

Since $M \subseteq M \cup K$, then M is not maximal.

Thus M and K are disjoint.

Since M and K are distinct μ^*-components of X.

By Theorem 3.1.23, we get that M and K are maximal μ^{*s}-connected sets of X.

So, $M = X \setminus K$.

Thus $M \cup K = (X \setminus K) \cup K = X$. □

Theorem 3.1.32. Each μ^*-component of an ideal generalized topological space (X, μ, I) is μ^*-closed.

Proof. Let A be μ^*-component of X.

By Theorem 3.1.23, and Corollary 3.1.20, A is maximal μ^{*s}-connected and $c^{*\mu}(A)$ is μ^{*s}-connected.

Thus $A = c^{*\mu}(A)$.

This implies that A is μ^*-closed. □

CHAPTER 4

A NEW COLLECTION WHICH CONTAIN THE GENERALIZED TOPOLOGICAL SPACES

4.1 ξ-I-μ-open sets

This section discusses basic properties of ξ-I-μ-open sets and basic concepts type of generalized open sets with ideal.

Definition 4.1.1. Let G be a subset of an ideal generalized topological space (X, μ, I). Then G is said to be

(1) strongly β-I-μ-open if $G \subseteq c^{*\mu}(i_\mu(c^{*\mu}(G)))$.

(2) semi-I-μ-open if $G \subseteq c^{*\mu}(i_\mu(G))$.

(3) pre-I-μ-open if $G \subseteq i_\mu(c^{*\mu}(G))$.

(4) pre-I-μ-closed if $X \setminus G$ is pre-I-μ-open.

Definition 4.1.2. Let G be a subset of an ideal generalized topological space (X, μ, I). Then G is said to be β-I-μ-open if $G \subseteq c_\mu(i_\mu(c^{*\mu}(G)))$.

The complement of a β-I-μ-open set is called β-I-μ-closed. The family of all β-I-μ-open (resp. β-I-μ-closed) sets of (X, μ, I) are denoted by $\beta I\mu O(X)$ (resp.$\beta I\mu C(X)$. The family of all β-I-μ-open (resp. β-I-μ-closed) sets of (X, μ, I) containing a point $x \in X$ are denoted by $\beta I\mu O(X, x)$ (resp. $\beta I\mu C(X, x)$). The intersection of all β-I-μ-closed sets containing G are called the β-I-μ-closure of G and are denoted by $Cl_{\beta I\mu}(G)$. The β-I-μ-interior of G is defined by the union of all β-I-μ-open sets contained in G and is denoted by $Int_{\beta I\mu}(G)$.

Definition 4.1.3. Let G be a subset of an ideal generalized topological space (X, μ, I). Then G is said to be

(1) $semi^*$-I-μ-open if $G \subseteq c_\mu(i^{*\mu}(G))$.

(2) $semi^*$-I-μ-closed if $X \setminus G$ is $semi^*$-I-μ-open.

Definition 4.1.4. Let G be a subset of an ideal generalized topological space (X, μ, I). Then G is called ξ-I-μ-open set if $G = \emptyset$ or there exists a nonempty pre-I-μ-open subset K such that $K \setminus c^{*\mu}(G) \in I$.

The complement of a ξ-I-μ-open set is called ξ-I-μ-closed. The family of all ξ-I-μ-open (resp. ξ-I-μ-closed) sets of (X, μ, I) is denoted by $O_\xi(X)$ (resp. $C_\xi(X)$). The family of all ξ-I-μ-open (resp. ξ-I-μ-closed) sets of (X, μ, I) containing a point $x \in X$ is denoted by $O_\xi(X, x)$ (resp. $C_\xi(X, x)$).

Theorem 4.1.5. Let G be a subset of an ideal generalized topological space (X, μ, I). For ξ-I-μ-open subset G_α in X for each $\alpha \in \Lambda$, $\bigcup\{G_\alpha; \alpha \in \Lambda\}$ is a ξ-I-μ-open subset of X.

Proof. Let G be a subset of an ideal generalized topological space (X, μ, I) and G_α is a ξ-I-μ-open subset of X, for each $\alpha \in \Lambda$.
If $\bigcup\{G_\alpha; \alpha \in \Lambda\} = \emptyset$, then $\bigcup\{G_\alpha; \alpha \in \Lambda\}$ is a ξ-I-μ-open subset of X.
In case $\bigcup\{G_\alpha; \alpha \in \Lambda\} \neq \emptyset$, then there exists an $\alpha_i \in \Lambda$ such that $G_{\alpha_i} \neq \emptyset$.
Since G_{α_i} is ξ-I-μ-open, then there exists a nonempty pre-I-μ-open subset K of X such that $K \setminus c^{*\mu}(G_{\alpha_i}) \in I$.
Since $G_{\alpha_i} \subseteq \bigcup\{G_\alpha; \alpha \in \Lambda\}$, then $c^{*\mu}(G_{\alpha_i}) \subseteq c^{*\mu}(\bigcup\{G_\alpha; \alpha \in \Lambda\})$.
So $K \setminus c^{*\mu}(\bigcup\{G_\alpha; \alpha \in \Lambda\}) \subseteq K \setminus c^{*\mu}(G_{\alpha_i})$.
Since $K \setminus c^{*\mu}(G_{\alpha_i}) \in I$, then $K \setminus c^{*\mu}(\bigcup\{G_\alpha; \alpha \in \Lambda\}) \in I$.
Thus $\bigcup\{G_\alpha; \alpha \in I\}$ is a ξ-I-μ-open subset of X. □

Lemma 4.1.6. Every μ-open is pre-I-μ-open.

Proof. Since H is μ-open i.e, $H = i_\mu(H)$, we have $H \subseteq i_\mu(H \cup H^*) = i_\mu(c^{*\mu}(H))$.
So H is pre-I-μ-open subset. □

Definition 4.1.7. Let G be a subset of an ideal generalized topological space (X, μ, I). The intersection of all ξ-I-μ-closed sets containing G is called the ξ-I-μ-closure of G and is denoted by $Cl_\xi(G)$.
The ξ-I-μ-interior of G is defined by the union of all ξ-I-μ-open sets contained in G and is denoted by $Int_\xi(G)$.

Lemma 4.1.8. Let G be a subset of an ideal generalized topological space (X, μ, I). Then $Cl_\xi(G) = G$ if and only if G is ξ-I-μ-closed .

Proof. Since $Cl_\xi(G) = \bigcap F_\alpha$ where F_α is ξ-I-μ-closed and $G \subseteq F_\alpha$.

So, $X \setminus F_\alpha$ is ξ-I-μ-open and $X \setminus F \subseteq X \setminus G$.

It follows that from Theorem 4.1.5, that $\bigcup_{\alpha\in\Lambda}(X \setminus F_\alpha)$ is ξ-I-μ-open.

Therefore $\bigcup_{\alpha\in\Lambda}(X \setminus F_\alpha) = X \setminus \bigcap_{\alpha\in\Lambda}(F_\alpha)$.

Hence $\bigcap_{\alpha\in\Lambda}(F_\alpha)$ is ξ-I-μ-closed.

Conversely, it is obvious. $\square$

Lemma 4.1.9. Let (X, μ, I) be an ideal generalized topological space, then $c^{*\mu}(i_\mu(X)) = X$.

Proof. It is obvious that $c^{*\mu}(i_\mu(X)) \subseteq X$.

Consider $X \in \mu$, we get that $X = i_\mu(X) \subseteq i_\mu(X) \cup (i_\mu(X))^* = c^{*\mu}(i_\mu(X))$.

Assume that there exists $y \in X$ such that $y \notin c^{*\mu}(i_\mu(X))$.

Then we have $y \notin i_\mu(X)$ and $y \notin (i_\mu(X))^*$.

Since $y \notin (i_\mu(X))^*$, there exists $G \in \mu(y)$ such that $G \cap (i_\mu(X)) \in I$.

Thus $y \in G \subseteq \bigcup\{G : G \in \mu, G \subseteq X\} = i_\mu(X)$. This is a contradiction.

Therefore, $c^{*\mu}(i_\mu(X)) = X$. $\square$

Theorem 4.1.10. Let G be a subset of an ideal generalized topological space (X, μ, I). If G is a strongly β-I-μ-open subset of X, then G is a ξ-I-μ-open subset of X.

Proof. Let G be a subset of an ideal generalized topological space (X, μ, I) and G be a strongly β-I-μ-open subset of X.

Case 1 : For $G = \emptyset$, it is clearly that $\emptyset$ is ξ-I-μ-open.

Case 2 : For G is a nonempty subset in X. Then we get that $G \subseteq c^{*\mu}(i_\mu(c^{*\mu}(G)))$.

Assume that $K = i_\mu(c^{*\mu}(G))$. Then $K \subseteq c^{*\mu}(G)$.

Since $i_\mu(c^{*\mu}(G))$ is μ-open, it follows that K is μ-open.

By Lemma 4.1.6, we have K is pre-I-μ-open.

Next, we will to show that $K = i_\mu(c^{*\mu}(G))$ is a nonempty subset.

Suppose that $K = i_\mu(c^{*\mu}(G)) = \emptyset$.

As $G \subseteq c^{*\mu}(i_\mu(c^{*\mu}(G))) = c^{*\mu}(\emptyset) = \emptyset$. This is a contradiction.

Thus K is a nonempty subset.

And since $K \subseteq c^{*\mu}(G)$ and $\emptyset \in I$, it follows that $K \setminus c^{*\mu}(G) = \emptyset \in I$.

Therefore G is a ξ-I-μ-open subset of X. $\square$

Example 4.1.11. Let $X = \{a, b, c, d\}$. $\mu = \{\emptyset, X, \{a\}, \{a, b\}, \{c, d\}, \{a, c, d\}\}$ and $I = \{\emptyset, \{a\}, \{d\}, \{a, d\}\}$.

Take $G = \{d\} \subseteq X$. We get that $G^{*\mu} = \emptyset$ and $c^{*\mu}(G) = G \cup G^{*\mu} = \{d\}$.

Thus $i_\mu(c^{*\mu}(G) = \emptyset$ and $c^{*\mu}(i_\mu(c^{*\mu}(G)) = \emptyset$

So $G \nsubseteq c^{*\mu}(i_\mu(c^{*\mu}(G))$

Since $\{a\}$ is pre-I-μ-open, it follows that $\{a\} \setminus \{d\} = \{a\} \in I$.

Hence $G = \{d\}$ is ξ-I-μ-open.

Therefore G is a ξ-I-μ-open subset of X and G is not a strongly β-I-μ-open subset of X.

Theorem 4.1.12. Let (X, μ, I) be an ideal generalized topological space. Each μ^*-dense subset of X is ξ-I-μ-open.

Proof. Assume that G be a μ^*-dense, then $c^{*\mu}(G) = X$.

And we get that $G \subseteq c^{*\mu}(G) = X = c^{*\mu}(i_\mu(X)) = c^{*\mu}(i_\mu(c^{*\mu}(G)))$.

So G is a strongly β-I-μ-open set in X.

By Theorem 4.1.10, we get that G is ξ-I-μ-open. □

Definition 4.1.13. Let G be a subset of an ideal generalized topological space (X, μ, I). Then G is called μ^*-nowhere dense if $i_\mu(c^{*\mu}(G)) = \emptyset$.

Theorem 4.1.14. Let G be a nonempty subset of ideal generalized topological space (X, μ, I). If G is not a μ^*-nowhere dense set, then G is a ξ-I-μ-open.

Proof. Let G be a nonempty subset of ideal generalized topological space (X, μ, I).

Suppose that G is not a μ^*-nowhere dense set. Then $i_\mu(c^{*\mu}(G)) \neq \emptyset$.

Let $K = i_\mu(c^{*\mu}(G))$, we have $K \subseteq c^{*\mu}(G)$ and $i_\mu(K) = i_\mu(i_\mu(c^{*\mu}(G))) = i_\mu(c^{*\mu}(G)) = K$.

So, $K = i_\mu(K) \subseteq i_\mu(K \cup K^*) = i_\mu(c^{*\mu}(K))$. Thus K is pre-I-μ-open.

Since $K \subseteq c^{*\mu}(G)$, then $K \setminus c^{*\mu}(G) = \emptyset \in I$.

Therefore G is a ξ-I-μ-open set. □

Remark 4.1.15. The following example shows that we have a nonempty, ξ-I-μ-open and μ^*-nowhere dense set.

Example 4.1.16. Let $X = \{a, b, c, d\}$. $\mu = \{\emptyset, X, \{a\}, \{a, b\}, \{c, d\}, \{a, c, d\}\}$ and $I = \{\emptyset, \{a\}, \{d\}, \{a, d\}\}$.

Take $G = \{d\} \subseteq X$. We get that $G^{*\mu} = \emptyset$ and $c^{*\mu}(G) = G \cup G^{*\mu} = \{d\}$.

Thus $i_\mu(c^{*\mu}(G) = \emptyset$. So G is μ^*-nowhere dense.

Since $\{a\}$ is pre-I-μ-open, it follows that $\{a\} \setminus \{d\} = \{a\} \in I$.

Hence $G = \{d\}$ is ξ-I-μ-open.

Therefore G is a ξ-I-μ-open subset of X and μ^*-nowhere dense subset of X.

Definition 4.1.17. A function $f : (X, \mu, I) \to (Y, \nu)$ is said to be strongly β-I-μ-continuous if for each $x \in X$ and each ν-open set K of Y containing $f(x)$, there exists $M \in \beta I\mu O(X, x)$ such that $f(Cl_{\beta I\mu}(M)) \subseteq c_\nu(K)$.

Definition 4.1.18. A function $f : (X, \mu, I) \to (Y, \nu)$ is said to be ξ-I-μ-continuous at a point $x \in X$ if for each ν-open set K of Y containing $f(x)$, there exists $M \in O_\xi(X, x)$ such that $f(Cl_\xi(M)) \subseteq c_\nu(K)$.

Theorem 4.1.19. Every strongly β-I-μ-continuous is ξ-I-μ-continuous.

Proof. Let $f : (X, \mu, I) \to (Y, \nu)$ be a strongly β-I-μ-continuous function on X.

Then for each $x \in X$ and K is a ν-open set of Y containing $f(x)$, there exists $M \in \beta I\mu O(X, x)$ such that $f(Cl_{\beta I\mu}(M)) \subseteq c_\nu(K)$.

Since K is ν-open, we have K is strongly-β-I-μ-open.

By Theorem 4.1.10, then K is ξ-I-μ-open.

Since $M \subseteq Cl_\xi(M)$ and $Cl_\xi(M) \subseteq Cl_{\beta I\mu}(M)$.

It follows that $f(Cl_\xi(M)) \subseteq f(Cl_{\beta I\mu}(M)) \subseteq c_\nu(K)$.

Hence $f(Cl_\xi(M)) \subseteq c_\nu(K)$.

Therefore f is ξ-I-μ-continuous. □

Theorem 4.1.20. For a function f : $(X, \mu, I) \to (Y, \nu)$, the following properties are equivalent :

1. f is ξ-I-μ-continuous.
2. $f^{-1}(K)$ is ξ-I-μ-open in X for each ν-open set K of Y.
3. $f^{-1}(F)$ is ξ-I-μ-closed in X for each ν-closed set F of Y.
4. $f(Cl_\xi(A)) \subseteq c_\nu(f(A))$ for each subset A of X.
5. $Cl_\xi(f^{-1}(B)) \subseteq f^{-1}(c_\nu(B))$ for each subset B of Y.

Proof. (1) $\to$ (2): Let K be any ν-open subset of Y and $x \in f^{-1}(K)$.
Then there exists $M_x \in O_\xi(X, x)$ such that $f(M_x) \subseteq f(Cl_\xi(M_x) \subseteq K$.
We get that $f^{-1}(f(M_x)) \subseteq f^{-1}(K)$. So $M_x \subseteq f^{-1}(K)$.
Thus $x \in M_x \subseteq f^{-1}(K)$ and $f^{-1}(K) = \bigcup_{x \in f^{-1}(K)} M_x$.
Since any union of ξ-I-μ-open sets is ξ-I-μ-open, we get that $f^{-1}(K)$ is ξ-I-μ-open in X.

(2) $\to$ (3): Let F be ν-closed in Y, we get that $Y \setminus F$ is ν-open in Y.
By (2), $f^{-1}(Y \setminus F)$ is ξ-I-μ-open in X. But $f^{-1}(Y \setminus F) = X \setminus f^{-1}(F)$.
Therefore $f^{-1}(F)$ is ξ-I-μ-closed in X.

(3) $\to$ (4): Let A be any subset of X.
Assume $c_\nu(f(A))$ is ν-closed in Y and by (3), $f^{-1}(c_\nu(f(A)))$ is ξ-I-μ-closed.
Thus $Cl_\xi(A) \subseteq Cl_\xi(f^{-1}(f(A))) \subseteq Cl_\xi(f^{-1}(c_\nu(f(A)))) = f^{-1}(c_\nu(f(A)))$.
Hence $f(Cl_\xi(A)) \subseteq f(f^{-1}(c_\nu(f(A)))) \subseteq c_\nu(f(A))$.
Therefore $f(Cl_\xi(A)) \subseteq c_\nu(f(A))$.

(4) $\to$ (5): Let B be any subset of Y.
By (4), $f(Cl_\xi(f^{-1}(B))) \subseteq c_\nu(f(f^{-1}(B))) \subseteq c_\nu(B)$.
Thus $Cl_\xi(f^{-1}(B)) \subseteq f^{-1}(f(Cl_\xi(f^{-1}(B)))) \subseteq f^{-1}(c_\nu(B))$.
Hence $Cl_\xi(f^{-1}(B)) \subseteq f^{-1}(c_\nu(B))$.

(5) $\to$ (1): Let $x \in X$ and K be any ν-open set of Y containing $f(x)$.
Thus $Y \setminus K$ is ν-closed in Y,

$$\begin{aligned} Cl_\xi(f^{-1}(Y \setminus K)) &\subseteq f^{-1}(c_\nu(Y \setminus K)) \\ &= f^{-1}(Y \setminus i_\nu(K)) \\ &= f^{-1}(Y \setminus K) \\ &\subseteq Cl_\xi(f^{-1}(Y \setminus K)) \end{aligned}$$

It follow from Lemma 4.18 that $f^{-1}(Y \setminus K)$ is ξ-I-μ-closed in X and $f^{-1}(K)$ is a ξ-I-μ-open set of X containing x.
So, there exists $M \in O_\xi(X, x)$ such that $Cl_\xi(M) \subseteq f^{-1}(K)$.
Therefore $f(M) \subseteq f(Cl_\xi(M)) \subseteq f(f^{-1}(K)) \subseteq K \subseteq c_\nu(K)$.
This implies that f is ξ-I-μ-continuous. □

Theorem 4.1.21. Let G be a subset of an ideal generalized topological space (X, μ, I). Then G is a ξ-I-μ-open set in X if and only if $G = \emptyset$ or there exist a set H in I and a nonempty pre-I-μ-open set K such that $K \setminus H \subseteq c^{*\mu}(G)$.

Proof. Let (X, μ, I) be an ideal generalized topological space and $G \subseteq X$.

$(\Rightarrow)$ Assume that G is a ξ-I-μ-open set in X.

If $G \neq \emptyset$, then there exists a nonempty pre-I-μ-open set K such that $K \setminus c^{*\mu}(G) \in I$.

Assume that $H = K \setminus c^{*\mu}(G)$.

Then $K \setminus H = K \setminus (K \setminus c^{*\mu}(G))$

$= K \cap (K \cap c^{*\mu}(G)^c)^c$

$= K \cap (K^c \cup c^{*\mu}(G))$

$= (K \cap K^c) \cup (K \cap c^{*\mu}(G))$

$= K \cap c^{*\mu}(G)$

$\subseteq c^{*\mu}(G)$.

Thus $K \setminus H \subseteq c^{*\mu}(G)$.

$(\Leftarrow)$ Assume that $G = \emptyset$ or there exist a set $H \in I$ and a nonempty pre-I-μ-open set K such that $K \setminus H \subseteq c^{*\mu}(G)$.

If $G = \emptyset$, then G is a ξ-I-μ-open set in X .

If $G \neq \emptyset$, then there exists a set $H \in I$ and nonempty pre-I-μ-open set K such that $K \setminus H \subseteq c^{*\mu}(G)$.

Then $K \setminus c^{*\mu}(G) \subseteq K \setminus (K \setminus H) = K \cap H \subseteq H$.

Since $H \in I$, then $K \setminus c^{*\mu}(G) \in I$.

Therefore G is a ξ-I-μ-open set in X. $\square$

Theorem 4.1.22. Let G be a subset of an ideal generalized topological space (X, μ, I). Then G is ξ-I-μ-open in X if and only if

$G \in \{M \neq \emptyset :$ there exists a nonempty pre-I-μ-open subset K and a set $H \in I$ such that $K \subseteq c^{*\mu}(M) \cup H\} \cup \{\emptyset\} \subseteq P(X)$.

Proof. Let (X, μ, I) be an ideal generalized topological space and $G \subseteq X$.

$(\Rightarrow)$ Assume that G is a ξ-I-μ-open subset of X. Then $G = \emptyset$ or $G \neq \emptyset$.

Suppose that $G \neq \emptyset$.

By Theorem 4.1.21, there exists a set $H \in I$ and a nonempty pre-I-μ-open subset K such that $K \setminus H \subseteq c^{*\mu}(G)$.

And $(K \setminus H) \cup H = (K \cap H^c) \cup H = (K \cup H) \cap (H^c \cup H) = K \cup H$.

Then $K \subseteq K \cup H = (K \setminus H) \cup H \subseteq c^{*\mu}(G) \cup H$.

($\Leftarrow$) Assume that there exists an element $H \in I$ and nonempty pre-I-μ-open subset K such that $K \subseteq c^{*\mu}(G) \cup H$.

Then $K \setminus c^{*\mu}(G) \subseteq (c^{*\mu}(G) \cup H) \setminus c^{*\mu}(G) = (c^{*\mu}(G) \cap (c^{*\mu}(G))^c) \cup (H \cap c^{*\mu}(G)) = H \cap c^{*\mu}(G) \subseteq H$.

Since $H \in I$, then $K \setminus c^{*\mu}(G) \in I$.

Therefore G is a ξ-I-μ-open set in X. □

4.2 ξ-I-μ-closed sets and other properties

This section discusses about the collection of ξ-I-μ-closed sets in ideal generalized topological spaces. Properties for the collection of ξ-I-μ-closed sets and other properties for the collection of ξ-I-μ-closed sets are studied.

Definition 4.2.1. Let G be a subset of an ideal generalized topological space (X, μ, I). Then G is called ξ-I-μ-closed set if $X \setminus G$ is a ξ-I-μ-open subset of X.

Theorem 4.2.2. Let G be a subset of an ideal generalized topological space (X, μ, I). Then G is called ξ-I-μ-closed if and only if there exists $H \in I$ and a pre-I-μ-closed subset $L \neq X$ such that $i^{*\mu}(G) \setminus H \subseteq L$ or $G = X$.

Proof. Let G be a subset of an ideal generalized topological spaces (X, μ, I).

($\Rightarrow$) Assume that G be a ξ-I-μ-closed set.

Then $X \setminus G$ is a ξ-I-μ-open set.

By Theorem 4.1.22, we have $X \setminus G = \emptyset$ or there exists a set H in I and a nonempty pre-I-μ-open subset K such that $K \subseteq c^{*\mu}(X \setminus G) \cup H$.

In case, $X \setminus G = \emptyset$, we have $X = G$.

If $X \setminus G \neq \emptyset$, there exists a set H in I and a nonempty pre-I-μ-open set K such that $K \subseteq c^{*\mu}(X \setminus G) \cup H$.

This implies that
$$\begin{aligned} i^{*\mu}(G) \setminus H &= i^{*\mu}(G) \cap H^c \\ &= (X \setminus c^{*\mu}(X \setminus G) \cap (X \setminus H) \\ &= X \setminus (c^{*\mu}(X \setminus G) \cup H) \\ &\subset X \setminus K. \end{aligned}$$

Let $L = X \setminus K$.

Since K is a nonempty pre-I-μ-open subset, we have $X \setminus K$ is a nonempty pre-I-μ-closed subset and $X \setminus K \neq X$ So $L \neq X$.

Thus $i^{*\mu}(G) \setminus H \subseteq L$.

($\Leftarrow$) Let G be a subset of X.

If $X = G$, then $X \setminus G = \emptyset$. So $X \setminus G$ is ξ-I-μ-open.

Thus G is ξ-I-μ-closed.

If $X \neq G$, there exist $H \in I$ and a pre-I-μ-closed subset $K \neq X$ such that $i^{*\mu}(G) \setminus H \subseteq K$.

$$
\begin{aligned}
\text{So } X \setminus K \subseteq X \setminus (i^{*\mu}(G) \setminus H) &= X \setminus (i^{*\mu}(G) \cap (X \setminus H)) \\
&= (X \setminus i^{*\mu}(G)) \cup (X \setminus (X \setminus H)) \\
&= c^{*\mu}(X \setminus G) \cup H
\end{aligned}
$$

Thus $X \setminus K \subseteq c^{*\mu}(X \setminus G) \cup H$.

By Theorem 4.1.22, we get that $X \setminus G$ is ξ-I-μ-open.

Hence G is a ξ-I-μ-closed subset. $\square$

Theorem 4.2.3. Let G be a subset of an ideal generalized topological space (X, μ, I). Then G is ξ-I-μ-closed if and only if there exists a pre-I-μ-closed subset $K \neq X$ such that $i^{*\mu}(G) \setminus K \in I$ or $G = X$.

Proof. Let G be a subset of an ideal generalized topological space (X, μ, I).

($\Rightarrow$) Suppose that G is a ξ-I-μ-closed set.

By Theorem 4.2.2, there exists $H \in I$ and a pre-I-μ-closed subset K and $K \neq X$ such that $i^{*\mu}(G) \setminus H \subseteq K$ or $G = X$.

In case $i^{*\mu}(G) \setminus H \subseteq K$, we get that

$$i^{*\mu}(G) \setminus K \subseteq i^{*\mu}(G) \setminus (i^{*\mu}(G) \setminus H) = i^{*\mu}(G) \cap H \subseteq H.$$

So $i^{*\mu}(G) \setminus K \subseteq H$.

Since $H \in I$, this implies that $i^{*\mu}(G) \setminus K \in I$.

($\Leftarrow$) Assume that there exists $K \neq X$ is a pre-I-μ-closed subset with $i^{*\mu}(G) \setminus K \in I$ or $G = X$.

We have $i^{*\mu}(G) \setminus K = i^{*\mu}(G) \setminus K$, and also

$$i^{*\mu}(G) \setminus (i^{*\mu}(G) \setminus K) \subseteq i^{*\mu}(G) \setminus (i^{*\mu}(G) \setminus K) = K.$$

By Theorem 4.2.2, G is a ξ-I-μ-closed subset of X. □

Lemma 4.2.4. Let (X, μ, I) be an ideal generalized topological space.

(1) If $M \in \mu$ and $M \cap A \in I$ implies that $M \cap A^* = \emptyset$.

(2) $(A \cup A^*)^* \subseteq A^*$, for $A \subseteq X$.

(3) $c^{*\mu}(c^{*\mu}(A)) = c^{*\mu}(A)$, for $A \subseteq X$.

Proof. (1). Assume $x \in M \cap A^*$ i.e, $x \in M$ and $x \in A^*$.

It follows that $M \cap A \notin I$, for all $M \in \mu(x)$.

This is a contradiction.

Hence $M \cap A^* = \emptyset$.

(2) Assume that $x \notin A^*$.

There exists $M \in \mu(x)$ such that $M \cap A \in I$.

By (1), we have that $M \cap A^* = \emptyset$.

Thus $M \cap (A \cup A^*) = (M \cap A) \cup (M \cap A^*) = (M \cap A) \cup \emptyset = M \cap A \in I$.

Therefore $x \notin (A \cup A^*)^*$.

(3) It is obvious that $c^{*\mu}(A) \subseteq c^{*\mu}(c^{*\mu}(A))$.

By (2), we have $c^{*\mu}(c^{*\mu}(A)) = c^{*\mu}(A) \cup (c^{*\mu}(A))^* = (A \cup A^*) \cup (A \cup A^*)^* \subseteq (A \cup A^*) \cup A^* = c^{*\mu}(A)$.

Hence $c^{*\mu}(c^{*\mu}(A)) \subseteq c^{*\mu}(A)$. □

Theorem 4.2.5. Let G be a subset of an ideal generalized topological space (X, μ, I). Assume that every μ^*-open subset of X is pre-I-μ-closed. Then each subset of X is ξ-I-μ-open.

Proof. Let G be a subset of an ideal generalized topological space (X, μ, I).

Assume that every μ^*-open subset of X is pre-I-closed.

If $G = \emptyset$, then G is ξ-I-μ-open.

Let G be a nonempty subset of X.

We will show that $X \setminus c^{*\mu}(G)$ is μ^*-open. We have $c^{*\mu}(X \setminus (X \setminus c^{*\mu}(G)) = c^{*\mu}(c^{*\mu}(G)) = c^{*\mu}(G) = X \setminus (X \setminus c^{*\mu}(G))$.

Therefore $X \setminus c^{*\mu}(G)$ is μ^*-open. i.e, $c^{*\mu}(G)$ is μ^*-closed.

By the assumption $c^{*\mu}(G)$ is pre-I-μ-open.

Thus $c^{*\mu}(G) \setminus c^{*\mu}(G) = \emptyset \in I$.

Consequently, G is ξ-I-μ-open. $\square$

Remark 4.2.6. The following example shows that the intersection of two sets need not be ξ-I-μ-open set for any ideal generalized topological space (X, μ, I).

Example 4.2.7. Let $X = \{a, b, c, d\}$, $\mu = \{\emptyset, X, \{a\}, \{c\}, \{a, b\}, \{a, b, c\}, \{a, c, d\}\}$ And $I = \{\emptyset, \{b\}\}$.

Since $A = \{a, d\}$ and $B = \{c, d\}$ are ξ-I-μ-open set.

Then $A \cap B = \{d\}$ but $\{d\} \notin \xi$-I-μ-open set.

Theorem 4.2.8. Let (X, μ, I) be an ideal generalized topological space. If there exists $a \in X$ such that $\{a\}$ is a pre-I-μ-open and $\{a\} \in I$. Then each subset of X is ξ-I-μ-open.

Proof. Let (X, μ, I) be an ideal generalized topological space and G be a nonempty subset of X.

Let $\{a\}$ be pre-I-μ-open and $\{a\} \in I$.

Then $\{a\} \setminus c^{*\mu}(G) = \{a\}$ or $\{a\} \setminus c^{*\mu}(G) = \emptyset$.

This implies that $\{a\} \setminus c^{*\mu}(G) = \{a\} \in I$ or $\{a\} \setminus c^{*\mu}(G) = \emptyset \in I$.

And $\{a\}$ is a pre-I-μ-open subset of X. Therefore G is ξ-I-μ-open. $\square$

Example 4.2.9. Let $X = \{a, b, c, d\}$ and $\mu = \{\emptyset, X, \{a, b\}, \{a, c\}, \{a, b, c\}\}$, $I = \{\emptyset, \{a\}, \{d\}, \{a, d\}\}$.

Suppose $A = \{a\}$, then $A^{*\mu} = \emptyset$ and $c^{*\mu}(A) = \{a\}$.

Thus $i_\mu(c^{*\mu}(A)) = \emptyset$ and $A \nsubseteq i_\mu(c^{*\mu}(A))$.

So A is not a pre-I-μ-open subset.

Let $G = \{b, c\}$. Then $G^{*\mu} = \{a, b, c, d\}$ and $c^{*\mu}(G) = \{a, b, c, d\}$.

Thus $A \setminus c^{*\mu}(G) = \emptyset$.

Therefore G is ξ-I-μ-open.

Theorem 4.2.10. Let (X, μ, I) be an ideal generalized topological space and $\emptyset \neq G \subseteq F \subseteq X$ and G be ξ-I-μ-open. Then F is ξ-I-μ-open.

Proof. Let (X, μ, I) be an ideal generalized topological space and $\emptyset \neq G \subseteq F \subseteq X$ and G be ξ-I-μ-open.

Then there exists a nonempty pre-I-μ-open subset K such that $K \setminus c^{*\mu}(G) \in I$.
Since $G \subseteq F$ and $c^{*\mu}(G) \subseteq c^{*\mu}(F)$, then $K \setminus c^{*\mu}(F) \subseteq K \setminus c^{*\mu}(G)$.
Since $K \setminus c^{*\mu}(G) \in I$, we get that $K \setminus c^{*\mu}(F) \in I$.
Hence F is a ξ-I-μ-open set. □

Theorem 4.2.11. Let (X, μ, I) be an ideal generalized topological space. Then $\{a\}$ is $semi^{*}$-I-μ-closed or $\{a\}$ is a ξ-I-μ-open subset of X for each $a \in X$.

Proof. We show that $\{a\}$ is a ξ-I-μ-open subset.
Assume that $\{a\}$ is not a $semi^{*}$-I-μ-closed set in X.
Then $X \setminus \{a\} \nsubseteq c_{\mu}(i^{*\mu}(X \setminus \{a\}))$.
Since $c_{\mu}(i^{*\mu}(X \setminus \{a\})) = c_{\mu}(X \setminus c^{*\mu}(\{a\})) = X \setminus (i_{\mu}(c^{*\mu}(\{a\}))$, we get that $X \setminus \{a\} \nsubseteq X \setminus (i_{\mu}(c^{*\mu}(\{a\}))$.
There exists $y \in X \setminus \{a\}$ but $y \notin X \setminus (i_{\mu}(c^{*\mu}(\{a\}))$.
This implies that $y \in (i_{\mu}(c^{*\mu}(\{a\}))$. Thus $i_{\mu}(c^{*\mu}(\{a\}) \neq \emptyset$ and $i_{\mu}(c^{*\mu}(\{a\}) = i_{\mu}(i_{\mu}(c^{*\mu}(\{a\})) \subseteq i_{\mu}(c^{*\mu}(i_{\mu}(c^{*\mu}(\{a\}))$.
Hence, $i_{\mu}(c^{*\mu}(\{a\}))$ is nonempty pre-I-μ-open and $i_{\mu}(c^{*\mu}(\{a\}) \setminus c^{*\mu}(\{a\}) = \emptyset \in I$.
So $\{a\}$ is a ξ-I-μ-open. □

CHAPTER 5

CONCLUSIONS

The aim of this thesis is to introduce the results of connected in ideal generalized topological space. And we study characterization of ideal generalized topological space. The results are as follows:

1) An ideal generalized topological space (X, μ, I) is called μ^*-connected if X cannot be written as the disjoint union of a nonempty μ-open set and a nonempty μ^*-open set.

2) Nonempty subsets M, K of an ideal generalized topological space (X, μ, I) are called μ^*-separated if $c^{*\mu}(M) \cap K = M \cap c_\mu(K) = \emptyset$.

   From the above definitions, I have the following theorems are derived

   2.1) Let (X, μ, I) be an ideal generalized topological space. If M and K are μ^*-separated sets of X and $M \cup K \in \mu$, then M and K are μ-open and μ^*-open, respectively.

3) A subset M of an ideal generalized topological space (X, μ, I) is called μ^{*s}-connected if M is not the union of two μ^*-separated sets in (X, μ, I).

   From the above definitions, I have the following theorems are derived

   3.1) If M is a μ^{*s}-connected in X and H, K are μ^*-separated sets in X with $M \subseteq H \cup K$, then either $M \subseteq H$ or $M \subseteq K$.

   3.2) If M is a μ^{*s}-connected set of an ideal generalized topological space (X, μ, I) and $M \subseteq N \subseteq c^{*\mu}(M)$, then N is μ^{*s}-connected.

   3.3) If M is a μ^{*s}-connected set of an ideal generalized topological space (X, μ, I), then $c^{*\mu}(M)$ is μ^{*s}-connected.

   3.4) If $\{M_n : N \in \Lambda\}$ is a nonempty family of μ^{*s}-connected sets with $\bigcap_{n\in\Lambda} M_n \neq \emptyset$, then $\bigcup_{n\in\Lambda} M_n$ is a μ^{*s}-connected sets.

4) Let (X, μ, I) be an ideal generalized topological space and $x \in X$. The union of all μ^{*s}-connected subsets of X containing x is called the μ^*-component of X containing x.
From the above definitions, I have the following theorems are derived

4.1) Each μ^*-component of an ideal generalized topological space (X, μ, I) is a maximal μ^{*s}-connected.

4.2) The set of all distinct μ^*-component of an ideal generalized topological space (X, μ, I) forms a partition of X.

4.3) Each μ^*-component of an ideal generalized topological space (X, μ, I) is μ^*-closed.

5) Let G be a subset of an ideal generalized topological space (X, μ, I). Then G is called ξ-I-μ-open if $G = \emptyset$ or there exists a nonempty pre-I-μ-open subset K such that $K \setminus c^{*\mu}(G) \in I$.
The complement of a ξ-I-μ-open set is called ξ-I-μ-closed. The family of all ξ-I-μ-open (resp. ξ-I-μ-closed) sets of (X, μ, I) is denoted by $O_\xi(X)$ (resp. $C_\xi(X)$). The family of all ξ-I-μ-open (resp. ξ-I-μ-closed) sets of (X, μ, I) containing a point $x \in X$ is denoted by $O_\xi(X, x)$ (resp. $C_\xi(X, x)$).

6) If G_α is ξ-I-μ-open in X, for all $\alpha \in \Lambda$, then $\bigcup_{\alpha \in \Lambda} G_\alpha$ is ξ-I-μ-open.

7) Let G be a subset of an ideal generalized topological space (X, μ, I). The intersection of all ξ-I-μ-closed sets containing G is called the ξ-I-μ-closure of G and is denoted by $Cl_\xi(G)$.
The ξ-I-μ-interior of G is defined by the union of all ξ-I-μ-open sets contained in G and is denoted by $Int_\xi(G)$.
From the above definitions, I have the following theorems are derived

7.1) $Cl_\xi(G) = G$ if and only if G is ξ-I-μ-closed .

7.2) Let X is a subset of ideal generalized topological spaces, then $c^{*\mu}(i_\mu(X)) = X$.

8) Let G be a subset of an ideal generalized topological space (X, μ, I). Then G is called μ^*-nowhere dense if $i_\mu(c^{*\mu}(G)) = \emptyset$.

9) A function f : $(X, \mu, I) \rightarrow (Y, \sigma)$ is said to be ξ-I-μ-continuous at a point $x \in X$ if for each σ-open set K of Y containing $f(x)$, there exists $M \subseteq O_\xi(X, x)$ such that $f(Cl_\xi(M)) \subseteq K$.

10) Every strongly β-I-μ-continuous is ξ-I-μ-continuous.

11) For a function f : $(X, \mu, I) \rightarrow (Y, \sigma)$, the following properties are equivalent :

    1. f is ξ-I-μ-continuous.
    2. $f^{-1}(K)$ is ξ-I-μ-open in X for each σ-open set K of Y.
    3. $f^{-1}(F)$ is ξ-I-μ-closed in X for each σ-closed set F of Y.
    4. $f(Cl_\xi(A)) \subseteq c_\sigma(f(A))$ for each subset A of X.
    5. $Cl_\xi(f^{-1}(B)) \subseteq f^{-1}(c_\sigma(B))$ for each subset B of Y.

12) Let (X, μ, I) be an ideal generalized topological space and $G \subseteq X$. Then G is a ξ-I-μ-open in X if and only if $G = \emptyset$ or there exist a set H in I and a nonempty pre-I-μ-open set K such that $K \setminus H \subseteq c^{*\mu}(G)$.

13) Let G be a subset of an ideal generalized topological space (X, μ, I). Then G is called ξ-I-μ-closed set if $X \setminus G$ is a ξ-I-μ-open subset of X.

14) Let G be a subset of an ideal generalized topological space (X, μ, I). Then G is called ξ-I-μ-closed if and only if there exists $H \in I$ and a pre-I-μ-closed subset $L \neq X$ such that $i^{*\mu}(G) \setminus H \subseteq L$ or $G = X$.

15) Let (X, μ, I) be an ideal generalized topological space. Then $\{a\}$ is $semi^*$-I-μ-closed or $\{a\}$ is a ξ-I-μ-open subset of X for each $a \in X$.

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