ORDINARY SMOOTH r-MINIMAL STRUCTURES SPACES

ORATHAI SEEKUNSAN

A thesis submitted in partial fulfillment of the requirements for
the degree of Master of Science in Mathematics
at Mahasarakham University
January 2017

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The examining committee has unanimously approved this thesis, submitted by Miss Orathai Seekunsan, as a partial fulfillment of the requirements for the Master of Science in Mathematics at Mahasarakham University.

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Orathai Seekunsan

ชื่อเรื่อง ปริภูมิโครงสร้าง r เล็กสุดแบบเรียบสามัญ
ผู้วิจัย นางสาวอรทัย ศรีคูณแสน
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บทคัดย่อ

ในงานวิจัยนี้ ผู้วิจัยได้นำเสนอปริภูมิใหม่สองปริภูมิคือปริภูมิโครงสร้าง r เล็กสุดแบบเรียบสามัญ ซึ่งได้ศึกษาสมบัติของเซตเปิด เซตปิด ตัวดำเนินการปิดคลุม ตัวดำเนินการภายใน ความต่อเนื่องของฟังก์ชันและความกระชับ นอกจากนั้น เรายังได้ศึกษาเซตปิดวางนัยทั่วไปแบบ b ความสัมพันธ์แบบต่างๆ และลักษณะเฉพาะของ extremely disconnected, T_{gs} และอีกปริภูมิหนึ่งคือ ปริภูมิโครงสร้าง r วิภัชนัยอย่างอ่อน และศึกษาสมบัติบางประการของเซตเปิด เซตเปิด α เซตกึ่งเปิดและความต่อเนื่องของฟังก์ชันบางชนิด

คำสำคัญ : ปริภูมิโครงสร้าง r เล็กสุดแบบเรียบสามัญ; ฟังก์ชันต่อเนื่อง r-M; ความกระชับ r-OSM; เซตปิด r-mgb; extremely disconnected; ปริภูมิ T_{gs}; ปริภูมิโครงสร้าง r วิภัชนัยอย่างอ่อน; เซตเปิด α; ฟังก์ชันต่อเนื่อง α

TITLE Ordinary Smooth r-Minimal Structures Spaces

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ABSTRACT

In this thesis, we introduce two new spaces which is called ordinary smooth r-minimal structure spaces which study properties of open set, closed sets, closure operator, interior operator, continuity and compactness. Moreover, we also study b-generalized closed sets, their relationships and characterization of extremely disconnected and T_{gs} spaces. And other spaces is called fuzzy r-weak structure spaces. And study some properties of open sets, α-open sets, semiopen sets and some type of continuity.

Keywords : r-OSMS; r-M continuous; r-OSM compact; r-mgb closed sets; extremely disconnected; T_{gs} spaces; r-FWS; α-open sets; α-continuous.

CONTENTS

CHAPTER 1

INTRODUCTION

In 1965, Zadeh [11] introduced the concept of fuzzy set. Chang [1] define fuzzy topological space using fuzzy set. In [2, 9], Chattopadhyay, Hazra and Samanta introdued smooth fuzzy topological spaces which are a generalization of fuzzy topological space.

In 2009, Kim, Min and Yoo [3] introduced the concept of fuzzy r-minimal space which is an extension of the smooth fuzzy topological space and study fuzzy r-m continuity, fuzzy r-M open maps and fuzzy r-M closed maps. In 2009, Min and Kim [10] introduced the concept of fuzzy r-minimal compactness, almost fuzzy r-minimal compactness and nearly fuzzy r-minimal compactness on fuzzy r-minimal spaces and investigate the relationships between fuzzy r-M continuous mappings and such types of fuzzy r-minimal compactness. In 2010, Min [6] introduced the concept of a fuzzy weakly r-M continuous mapping on fuzzy r-minimal structure. After that, [8] notion of fuzzy almost r-M continuous mapping on fuzzy r-minimal structure and investigate and properties for mapping. The concept of fuzzy r-minimal α-open set on a fuzzy r-minimal space and some basic properties and also introduce the concepts of fuzzy r-M α-continuous and fuzzy r-$M(M^*)$ α-open mappings and characterization for such mappings by Min [7].

In 2012 [5], Lim, Ryoo and Hur introduce the concept of ordinary smooth topology on a set X the mapping $\tau : 2^X \to I$ satisfying three axioms, where 2 denotes the two points set $\{0, 1\}$ is called ordinary smooth topological spaces in short ost on X also studied some properties of ordinary smooth continuous. In [4], Lee, Lim and Hur redefined the notions of ordinary smooth closure and ordinary smooth interior. Also they introduced and studied some properties of compact in an ordinary smooth topological space, and redefine a new definitions of ordinary smooth closure and ordinary smooth interior.

For our purpose, we introduce the concepts of ordinary smooth r-minimal spaces which is an extension the concepts of open set, closed set, closure and interior it intersects on such. The studied properties of opens mapping, continuous mapping and compactness. And the study properties of α-open set, α-continuous in fuzzy r-weakly structure spaces.

In Chapter 1, is an introduction.

In Chapter 2, we presents some basic concepts and results of fuzzy r-minimal structure and ordinary smooth topology their proofs in the subsequent chapters.

In Chapter 3, we mention the concept of open set, closed set, closure and interior in ordinary smooth r-minimal spaces, study the relationships and ordinary smooth r-M continuous mappings and such types of ordinary smooth r-minimal compactness and introduced many relationships between some known types of generalized closed sets and r-mb generalized closed sets, Also we studied characterizations of extremely disconnected space and T_{gs} space on ordinary smooth r-minimal spaces.

In Chapter 4, we mention the concept of α-open set, α-continuity and α-open mappings in fuzzy r-weakly structure spaces.

In the last Chapter, is a conclusions.

CHAPTER 2

PRELIMINARIES

In this chapter, we will recall some definitions, notations, dealing with some preliminaries and some useful results that will be duplicated in later chapter.

2.1 Fuzzy Topological Spaces and Smooth Fuzzy Topological Spaces

Definition 2.1.1 [11] A *fuzzy set* on X is a mapping $\mu : X \to [0,1]$ and I^X will denote the family of all fuzzy sets in X.

Definition 2.1.2 [1] A *fuzzy point* x_α, $\alpha \in (0,1]$, is an element of I^X such that

$$x_\alpha(y) = \begin{cases} \alpha & \text{if } y = x \\ 0 & \text{if } y \neq x. \end{cases}$$

A fuzzy point $x_\alpha \in \mu$ iff $\alpha \leq \mu(x)$.

Let $X = \{x\}$ be a space of points. A fuzzy set A in X is characterized by a membership mapping $\mu_A(x)$ from X to the unit interval $[0,1]$.

Definition 2.1.3 [1] Let A and B be fuzzy sets in a spaces X, Then:

$$A = B \Leftrightarrow \mu_A(x) = \mu_B(x) \quad \text{for all } x \in X$$

$$A \subseteq B \Leftrightarrow \mu_A(x) \leq \mu_B(x) \quad \text{for all } x \in X$$

$$C = A \cup B \Leftrightarrow \mu_C(x) = \max[\mu_A(x), \mu_B(x)] \quad \text{for all } x \in X$$

$$D = A \cap B \Leftrightarrow \mu_D(x) = \min[\mu_A(x), \mu_B(x)] \quad \text{for all } x \in X$$

$$A' \Leftrightarrow 1 - \mu_A(x) \quad \text{for all } x \in X.$$

For a family of fuzzy sets, $A = \{A_i : i \in I\}$, the union, $\bigcup_{i \in I} A_i$, and the intersection, $\bigcap_{i \in I} A_i$, are defined by

$$\bigcup_{i \in I} A_i(x) = \sup_{i \in I}\{\mu_{A_i}(x)\} \quad \text{for all } x \in X$$

$$\bigcap_{i\in I} A_i(x) = \inf_{i\in I}\{\mu_{A_i}(x)\} \qquad \text{for all } x \in X.$$

Definition 2.1.4 [1] Let $f : X \to Y$ be a mapping, $\mu \in I^X$ and $\nu \in I^Y$. We define

$$f(\mu)(y) = \begin{cases} \sup\{\mu(x) : x \in f^{-1}(\{y\})\}, & \text{if } f^{-1}(\{y\}) \neq \emptyset, \\ 0 & \text{if } f^{-1}(\{y\}) \neq \emptyset. \end{cases}$$

and $f^{-1}(\nu)(x) = \nu(f(x))$ for all $x \in X$.

Definition 2.1.5 [1] A fuzzy topology is a family $\mathcal{T}$ of fuzzy sets in X which satisfies the following conditions:

1 $\emptyset, X \in \mathcal{T}$.

2 If $A, B \in \mathcal{T}$, then $A \cap B \in \mathcal{T}$.

3 If $A_i \in \mathcal{T}$ for each $i \in I$, then $\bigcup_{i\in I} A_i \in \mathcal{T}$.

The pair $(X, \mathcal{T})$ is a *fuzzy topological spaces*, or *fts* for short. Every member of $\mathcal{T}$ is called a $\mathcal{T}$*-open fuzzy set*. A fuzzy set is $\mathcal{T}$-closed if and only if its complement is $\mathcal{T}$-open.

Let I be the unit interval [0,1] of the real line. A member μ of I^X is called a fuzzy set of X. By $\widetilde{0}$ and $\widetilde{1}$,we denote constant maps on X with value 0 and 1, respectively. For any $\mu \in I^X$, μ^C denotes the comlement $\widetilde{1} - \mu$.

Definition 2.1.6 [9] A smooth fuzzy topology on X is a map $\tau : I^X \to I$ which satisfies the following properties:

1 $\tau(\widetilde{0}) = \tau(\widetilde{1}) = 1$.

2 $\tau(\mu_1 \cap \mu_2) \geq \tau(\mu_1) \wedge \tau(\mu_2)$.

3 $\tau(\cup\mu_i) \geq \wedge\tau(\mu_i)$ for each $i \in I$.

The pair (X, τ) is called a *smooth fuzzy topological spaces*.

2.2 Fuzzy r-Minimal Spaces and Fuzzy r-Minimal compactness

Definition 2.2.1 [10] Let X be a nonempty set and $r \in (0,1]$. A fuzzy family $\mathcal{M} : I^X \to I$ on X is said to have a fuzzy r-minimal structure if the family

$$\mathcal{M}_r = \{A \in I^X : \mathcal{M}(A) \geq r\}$$

contains $\widetilde{0}$ and $\widetilde{1}$.

Then the $(X, \mathcal{M})$ is called a *fuzzy r-minimal space* (simply, r-FMS). Every member of $\mathcal{M}_r$ is called a *fuzzy r-minimal open set*. A fuzzy set A is called a *fuzzy r-minimal closed set* if the complement of A (simply, A^C) is a fuzzy r-minimal open set.

Let $(X, \mathcal{M})$ be an r-FMS and $r \in (0,1]$. the *fuzzy r-minimal closure* and the *fuzzy r-minimal interior* of A, denoted by $mC(A,r)$ and $mI(A,r)$, respectively, are defined as

$$mC(A,r) = \cap\{B \in I^X : B^C \in \mathcal{M}_r \text{ and } A \subseteq B\},$$
$$mI(A,r) = \cup\{B \in I^X : B \in \mathcal{M}_r \text{ and } B \subseteq A\}.$$

Theorem 2.2.2 [10] Let $(X, \mathcal{M})$ be an r-FMS and A, B in I^X.

1 $mI(A,r) \subseteq A$ and if $A \in \mathcal{M}_r$, then $mI(A,r) = A$.

2 $A \subseteq mC(A,r)$ and if $A^C \in \mathcal{M}_r$, then $mC(A,r) = A$.

3 If $A \subseteq B$, then $mI(A,r) \subseteq mI(B,r)$ and $mC(A,r) \subseteq mC(B,r)$.

4 $mI(A,r) \cap mI(B,r) \supseteq mI(A \cap B, r)$ and $mC(A,r) \cup mC(B,r) \subseteq mC(A \cup B, r)$.

5 $mI(mI(A,r),r) = mI(A,r)$ and $mC(mC(A,r),r) = mC(A,r)$.

6 $\widetilde{1} - mC(A,r) = mI(\widetilde{1} - A, r)$ and $\widetilde{1} - mI(A,r) = mC(\widetilde{1} - A, r)$.

Definition 2.2.3 [10] Let $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ be a mapping on two r-FMS's. Then f is said to be

1 *fuzzy r-M continuous mapping* if for every $A \in \mathcal{N}_r$, $f^{-1}(A)$ is in $\mathcal{M}_r$,

2 *fuzzy r-M open mapping* if for every $A \in \mathcal{M}_r$, $f(A)$ is in $\mathcal{N}_r$.

Definition 2.2.4 [3] Let $(X, \mathcal{M})$ be an r-FMS and $\{A_i \in I^X : i \in J\}$. A is called a *fuzzy r-minimal cover* if $\cup\{A_i : i \in J\} = X$. It is a *fuzzy r-minimal open cover* if each A_i is a fuzzy r-minimal open set. A subcover of a fuzzy r-minimal cover A is a subfamily of it which also is a fuzzy r-minimal cover.

Definition 2.2.5 [3] Let $(X, \mathcal{M})$ be an r-FMS. A fuzzy set A of X is said to be *fuzzy r-minimal compact* if every fuzzy r-minimal open cover $\{A_i \in \mathcal{M}_r : i \in J\}$ of A has a finite subcover.

Theorem 2.2.6 [3] Let $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ be a fuzzy r-M continuous mapping on two r-FMS's. If A is a fuzzy r-minimal compact set, then $f(A)$ is also a fuzzy r-minimal compact set.

Definition 2.2.7 [3] Let $(X, \mathcal{M})$ be an r-FMS. A fuzzy set A in X is said to be *almost fuzzy r-minimal compact* if for every fuzzy r-minimal open cover $\{A_i \in I^X : i \in J\}$ of A, there exists $J_0 = \{j_1, j_2, ..., j_n\} \subseteq J$ such that $A \subseteq \bigcup_{i \in J_0} mC(A_i, r)$.

Theorem 2.2.8 [3] Let $(X, \mathcal{M})$ be an r-FMS. If a fuzzy set A in X is fuzzy r-minimal compact, then it is also almost fuzzy r-minimal compact.

Theorem 2.2.9 [10] Let $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ be a mapping on two r-FMS's. If

1 f is fuzzy r-M continuous.

2 $f^{-1}(B)$ is a fuzzy r-minimal closed set, for each fuzzy r-minimal closed set B in Y.

3 $f(mC(A, r)) \subseteq mC(f(A), r)$ for all $A \in I^X$.

4 $mC(f^{-1}(B), r) \subseteq f^{-1}(mC(B, r))$ for all $B \in I^Y$.

5 $f^{-1}(mI(B, r)) \subseteq mI(f^{-1}(B), r)$ for all $B \in I^Y$.

Then 1 $\Leftrightarrow$ 2 $\Rightarrow$ 3 $\Leftrightarrow$ 4 $\Leftrightarrow$ 5.

Theorem 2.2.10 [3] Let $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ be a fuzzy r-M continuous mapping on two r-FMS's. If A is an almost fuzzy r-minimal compact set, then $f(A)$ is also an almost fuzzy r-minimal compact set.

Definition 2.2.11 [3] Let $(X, \mathcal{M})$ be an r-FMS. A fuzzy set A in X is said to be *nearly fuzzy r-minimal compact* if for every fuzzy r-minimal open cover $\{A_i : i \in J\}$ of A, there exists $J_0 = \{j_1, j_2, ..., j_n\} \subseteq J$ such that $A \subseteq \bigcup_{i \in j_0} mI(mC(A_i, r), r)$.

Theorem 2.2.12 [3] Let $(X, \mathcal{M})$ be an r-FMS. If a fuzzy set A in X is a fuzzy r-minimal compact, then it is a nearly fuzzy r-minimal compact.

Theorem 2.2.13 [10] Let $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ be a mapping on two r-FMS's. If

1 f is fuzzy r-M open.

2 $f(mI(A, r)) \subseteq mI(f(A), r)$ for all $A \in I^X$.

3 $mI(f^{-1}(B), r) \subseteq f^{-1}(mI(B, r))$ for all $B \in I^Y$.
Then 1 $\Rightarrow$ 2 $\Leftrightarrow$ 3.

Theorem 2.2.14 [3] Let $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ be a fuzzy r-M continuous and fuzzy r-M open on two r-FMS's. If A is a nearly fuzzy r-minimal compact set, then $f(A)$ is a nearly fuzzy r-minimal compact set.

Definition 2.2.15 [3] Let $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ be a mapping on two r-FMS's. Then f is said to be *fuzzy weakly r-M continuous* if for fuzzy point x_α of X and each fuzzy r-minimal open set V containing $f(x_\alpha)$, there is a fuzzy r-minimal open set U containing x_α such that $f(U) \subseteq mC(V, r)$.

Theorem 2.2.16 [3] Let $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ be a fuzzy r-M continuous mapping on two r-FMS's. Then the following statements are equivalent:

1 f is fuzzy weakly r-M continuous.

2 $f^{-1}(V) \subseteq mI(f^{-1}(mC(V, r)), r)$ for each fuzzy r-minimal open set V in Y.

3 $mC(f^{-1}(B), r) \subseteq f^{-1}(mC(B, r))$ for each fuzzy r-minimal closed set B in Y.

4 $mC(f^{-1}(V), r) \subseteq f^{-1}(mC(V, r))$ for each fuzzy r-minimal open set V in Y.

Definition 2.2.17 [10] Let X be a nonempty set and $\mathcal{M} : I^X \to I$ a fuzzy family on X. Then fuzzy family $\mathcal{M}$ has the property $(\mathcal{U})$ if for $A_i \in \mathcal{M}(i \in J)$,

$$\mathcal{M}(\cup A_i) \geq \wedge \mathcal{M}(A_i).$$

Theorem 2.2.18 [10] Let $(X, \mathcal{M})$ be an r-FMS and $\mathcal{M}$ has the property $(\mathcal{U})$. Then

1 $mI(A, r) = A$ if and only if $A \in \mathcal{M}_r$ for all $A \in I^X$.

2 $mC(A, r) = A$ if and only if $A^C \in \mathcal{M}_r$ for all $A \in I^X$.

Theorem 2.2.19 [6] Let $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ be a mapping on two r-FMS's and $A \in I^Y$. If f is fuzzy weakly r-M continuous, then the following statements are hold:

1 $f^{-1}(A) \subseteq mI(f^{-1}(mC(A, r)), r)$ for all $A = mI(A, r)$.

2 $mC(f^{-1}(mI(A, r)), r) \subseteq f^{-1}(A)$ for all $A = mC(A, r)$.

Theorem 2.2.20 [6] Let $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ be a fuzzy weakly r-M continuous mapping on two r-FMS's. If A is a fuzzy r-minimal compact set in X and $\mathcal{M}$ has property $(\mathcal{U})$, then $f(A)$ is an almost fuzzy r-minimal compact set.

Theorem 2.2.21 [6] Let $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ be a fuzzy weakly r-M continuous and fuzzy r-M open mapping on two r-FMS's. If A is an almost fuzzy r-minimal compact set and $\mathcal{M}$ has property $(\mathcal{U})$, then $f(A)$ is an almost fuzzy r-minimal compact set.

Theorem 2.2.22 [6] Let $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ be a fuzzy weakly r-M continuous and fuzzy r-M open mapping on two r-FMS's. If A is a nearly fuzzy r-minimal compact set and $\mathcal{M}$ has property $(\mathcal{U})$, then $f(A)$ is a nearly fuzzy r-minimal compact set.

Definition 2.2.23 [8] Let $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ be a mapping on two r-FMS's. Then f is said to be *fuzzy almost r-M continuous* if for fuzzy point x_α of X and each fuzzy r-minimal open set V containing $f(x_\alpha)$, there is a fuzzy r-minimal open set U containing x_α such that $f(U) \subseteq mI(mC(V, r), r)$.

Theorem 2.2.24 [8] Let $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ be a mapping on two r-FMS's. Then the following statements are equivalent:

1 f is fuzzy almost r-M continuous.

2 $f^{-1}(B) \subseteq mI(f^{-1}(mI(mC(B, r), r)), r)$ for each fuzzy r-minimal open set B in Y.

3 $mC(f^{-1}(mC(mI(F,r),r)),r) \subseteq f^{-1}(F)$ for each fuzzy r-minimal closed set F in Y.

Theorem 2.2.25 [8] Let $f : (X,\mathcal{M}) \to (Y,\mathcal{N})$ be a fuzzy almost r-M continuous mapping on two r-FMS's. If A is a fuzzy r-minimal compact set in X and $\mathcal{M}$ has property $(\mathcal{U})$, then $f(A)$ is an nearly fuzzy r-minimal compact set.

Theorem 2.2.26 [8] Let $f : (X,\mathcal{M}) \to (Y,\mathcal{N})$ be a fuzzy r-M continuous and fuzzy r-M open mapping on two r-FMS's. If A is an almost fuzzy r-minimal compact set and $\mathcal{M}$ has property $(\mathcal{U})$, then $f(A)$ is an almost fuzzy r-minimal compact set.

Theorem 2.2.27 [8] Let $f : (X,\mathcal{M}) \to (Y,\mathcal{N})$ be a fuzzy almost r-M continuous and fuzzy r-M open mapping on two r-FMS's. If A is a nearly fuzzy r-minimal compact set and $\mathcal{M}$ has property $(\mathcal{U})$, then $f(A)$ is a nearly fuzzy r-minimal compact set.

2.3 Ordinary Smooth Topological Spaces

For any set X, let $2 = \{0,1\}$ and let 2^X denoted the set of all ordinary subsets of X. And union and intersections of ordinary subsets are denoted by $\wedge$ and $\vee$, respectively, and defined by

$$\vee A_i(x) = \sup\{A_i(x) : i \in J\}.$$

$$\wedge A_i(x) = \inf\{A_i(x) : i \in J\}.$$

Definition 2.3.1 [5] Let X be a nonempty set. Then a mapping $\tau : 2^X \to I$ is called an *ordinary smooth topology* (in short, *ost*) on X or a *gradation of openness of ordinary subsets of* X if satisfies the following axioms:

1 $\tau(\emptyset) = \tau(X) = 1$,

2 $\tau(A \cap B) \geq \tau(A) \wedge \tau(B)$ for all $A, B \in 2^X$,

3 $\tau(\bigcup_{\alpha\in\Gamma} A_\alpha) \geq \bigwedge_{\alpha\in\Gamma} \tau(A_\alpha)$ for all $\{A_\alpha\} \subseteq 2^X$.

The pair (X, τ) is called an *ordinary smooth topological space* (in short, *osts*). We denote the set of all ost's on X as $OST(X)$.

Definition 2.3.2 [4] Let (X, τ) be an osts and let $A \in 2^X$. Then *ordinary smooth closure* [resp. ordinary smooth interior] of A in X, denoted by $\overline{A}_r$ [resp. A°] is defined by

$$\overline{A} = \bigcap\{F \in 2^X : A \subseteq F \text{ and } \mathcal{C}_\tau(F) > 0\}$$

$$\text{[resp. } A^\circ = \bigcup\{U \in 2^X : U \subseteq A \text{ and } \tau(U) > 0\}\text{].}$$

Proposition 2.3.3 [4] Let (X, τ) be an osts and let $A, B \in 2^X$. Then:

1 If $A \subseteq B$, then $A^\circ \subseteq B^\circ$ and $\overline{A} \subseteq \overline{B}$,

2 $(A^\circ)^\circ = \overline{(A^C)}$,

3 $A^\circ = (\overline{(A^C)})^C$,

4 $\overline{A} = ((A^C)^\circ)^C$,

5 $(\overline{A})^C = (A^C)^\circ$.

Proposition 2.3.4 [4] Let (X, τ) be an osts and let $A, B \in 2^X$. Then:

1 $X^\circ = X$,

2 $A^\circ \subseteq A$,

3 $(A^\circ)^\circ = A^\circ$,

4 $(A \cap B)^\circ \subseteq A^\circ \cap B^\circ$.

Proposition 2.3.5 [4] Let (X, τ) be an osts and let $A, B \in 2^X$. Then:

1 $\overline{\emptyset} = \emptyset$,

2 $A \subseteq \overline{A}$,

3 $\overline{(\overline{A})} = \overline{A}$,

4 $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$.

Proposition 2.3.6 [4] Let (X, τ) be an osts and let $A, B \in 2^X$.

1 If $\tau(A) > 0$, then $A = A^\circ$.

2 If $\tau(A^C) > 0$, then $A = \overline{A}$.

3 If $r \in I_0$ such that $A = \overline{A_r}$, then $A = \overline{A}$.

4 If $r \in I_0$ such that $A = (A_r)^\circ$, then $A = A^\circ$.

Definition 2.3.7 [5] Let $f : (X, \tau_1) \to (Y, \tau_2)$ be a mapping on two osts's. Then f is said to be:

1 ordinary smooth continuous if $\tau_2(A) \leq \tau_1(f^{-1}(A))$, for all $A \in 2^X$.

2 ordinary smooth weakly continuous if for each $A \in 2^Y$, $\tau_2(A) > 0 \Rightarrow \tau_1(f^{-1}(A)) > 0$, for all $A \in 2^X$.

Proposition 2.3.8 [5] Let $f : (X, \tau_1) \to (Y, \tau_2)$ be a mapping on two osts's and let f be ordinary smooth weakly continuous. Then:

1 $f(\overline{A}) \subseteq \overline{f(A)}$, for all $A \in 2^X$.

2 $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$, for all $B \in 2^Y$.

3 $f^{-1}(B^\circ) \subseteq (f^{-1}(B))^\circ$, for all $B \in 2^Y$.

Corollary 2.3.9 [5] Let $f : (X, \tau_1) \to (Y, \tau_2)$ be a mapping on two osts's and let f be ordinary smooth continuous. Then:

1 $f(\overline{A}) \subseteq \overline{f(A)}$, for all $A \in 2^X$.

2 $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$, for all $B \in 2^X$.

3 $f^{-1}(B^\circ) \subseteq (f^{-1}(B))^\circ$, for all $B \in 2^X$.

Definition 2.3.10 [5] Let $f : (X, \tau_1) \to (Y, \tau_2)$ be a mapping on two osts's and let f is said to be:

1 *ordinary smooth open* if $\tau_1(A) \leq \tau_2(f(A))$, for all $A \in 2^X$.

2 *ordinary smooth closed* if $\tau_1(A^c) \leq \tau_2(f(A^c))$, for all $A \in 2^X$.

For an osts (X, τ), let us define $S(\tau) = \{A \in 2^X : \tau(A) > 0\}$ and $S(\tau)$ will be called the *support* of τ.

Definition 2.3.11 [4] Let (X, τ) be an osts. A subsetes A in X is said to be:

1 *ordinary smooth compact* if for every family $\{A_\alpha\}_{\alpha\in\Gamma}$ in $S(\tau)$ covering X, there is a finite subset Γ_0 of Γ such that $\bigcup_{\alpha\in\Gamma_0} A_\alpha = X$.

2 *ordinary smooth almost compact* if for every family $\{A_\alpha\}_{\alpha\in\Gamma}$ in $S(\tau)$ covering X, there is a finite subset Γ_0 of Γ such that $\bigcup_{\alpha\in\Gamma_0} \overline{A}_\alpha = X$.

3 *ordinary smooth nearly compact* if for every family $\{A_\alpha\}_{\alpha\in\Gamma}$ in $S(\tau)$ covering X, there is a finite subset Γ_0 of Γ such that $\bigcup_{\alpha\in\Gamma_0} (\overline{A}_\alpha)^\circ = X$.

Proposition 2.3.12 [4] Let $f : (X, \tau_1) \to (Y, \tau_2)$ be a mapping on two osts's and let f be surjective and ordinary smooth weakly continuous. If (X, τ_1) is ordinary smooth almost compact, then so is (Y, τ_2).

Corollary 2.3.13 [4] Let $f : (X, \tau_1) \to (Y, \tau_2)$ be a mapping on two osts's and let f be surjective and ordinary smooth weakly continuous. If (X, τ_1) is ordinary smooth nearly compact, then (Y, τ_2) is ordinary smooth almost compact.

2.4 Fuzzy r-Minimal α - open Sets on Fuzzy Minimal Spaces

Definition 2.4.1 [7] Let $(X, \mathcal{M})$ be an r-FMS and $A \in I^X$. Then a fuzzy set A is called a *fuzzy r-minimal semiopen set* in X if

$$A \subseteq mC(mI(A, r), r).$$

A fuzzy set A is called a *fuzzy r-minimal semiclosed set* if the complement of A is fuzzy r-minimal semiopen.

Definition 2.4.2 [7] Let $(X, \mathcal{M})$ be an r-FMS and $A \in I^X$. Then a fuzzy set A is called a *fuzzy r-minimal α-open set* in X if

$$A \subseteq mI(mC(mI(A, r), r), r).$$

A fuzzy set A is called a *fuzzy r-minimal α-closed set* if the complement of A is fuzzy r-minimal α-open.

Lemma 2.4.3 [7] Let $(X, \mathcal{M})$ be an r-FMS. Then a fuzzy set A is fuzzy r-minimal α-closed set if and only if $mC(mI(mC(A, r), r), r) \subseteq A$.

Theorem 2.4.4 [7] Let $(X, \mathcal{M})$ be an r-FMS. Then any union of fuzzy r-minimal α-open set is fuzzy r-minimal α-open.

Definition 2.4.5 [7] Let $(X, \mathcal{M})$ be an r-FMS. For any $A \in I^X$, $m\alpha C(A, r)$ and $m\alpha I(A, r)$, respectively, are defined as the follows

$$m\alpha C(A, r) = \cap\{F \in I^X : A \subseteq F, F \text{ is fuzzy } r\text{-minimal } \alpha\text{-closed }\};$$

$$m\alpha I(A, r) = \cup\{U \in I^X : U \subseteq A, U \text{ is fuzzy } r\text{-minimal } \alpha\text{-open }\}.$$

Theorem 2.4.6 [7] Let $(X, \mathcal{M})$ be an r-FMS and $A \in I^X$. Then the following statments are hold.

1 $m\alpha I(A, r) \subseteq A$.

2 If $A \subseteq B$, then $m\alpha I(A, r) \subseteq m\alpha I(B, r)$.

3 A is fuzzy r-minimal α-open if and only if $m\alpha I(A, r) = (A, r)$.

4 $m\alpha I(m\alpha I(A, r), r) = m\alpha I(A, r)$.

5 $m\alpha C(\widetilde{1} - A, r) = \widetilde{1} - m\alpha I(A, r)$ and $m\alpha I(\widetilde{1} - A, r) = \widetilde{1} - m\alpha C(A, r)$.

Theorem 2.4.7 [7] Let $(X, \mathcal{M})$ be an r-FMS and $A \in I^X$. Then

1 $A \subseteq m\alpha C(A, r)$.

2 If $A \subseteq B$, then $m\alpha C(A, r) \subseteq m\alpha C(B, r)$.

3 A is fuzzy r-minimal α-closed if and only if $m\alpha C(A, r) = (A, r)$.

4 $m\alpha C(m\alpha I(A, r), r) = m\alpha C(A, r)$.

2.5 Fuzzy r-M α-continuity and Fuzzy r-M α-open mappings

Definition 2.5.1 [7] Let $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ be a mapping on two r-FMS's. Then f is said to be *fuzzy r-M α-continuous* if for each point x_α and each fuzzy r-minimal open set V containing $f(x_\alpha)$, there exists a fuzzy r-minimal α-open set U containing x_α such that $f(U) \subseteq V$.

Definition 2.5.2 [7] Let $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ be a mapping on two r-FMS's. Then f is said to be *fuzzy r-M semicontinuous* if for each point x_α and each fuzzy r-minimal open set V containing $f(x_\alpha)$, there exists a fuzzy r-minimal semiopen set U containing x_α such that $f(U) \subseteq V$.

Theorem 2.5.3 [7] Let $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ be a mapping on two r-FMS's. Then the following statements are equivalent:

1 f is fuzzy r-M α-continuous.

2 $f^{-1}(V)$ is a fuzzy r-minimal α-open set for each fuzzy r-minimal open set V in Y.

3 $f^{-1}(B)$ is a fuzzy r-minimal α-closed set for each fuzzy r-minimal closed set B in Y.

4 $f(m\alpha C(A, r)) \subseteq mC(f(A), r)$ for $A \in I^X$.

5 $m\alpha C(f^{-1}(B), r) \subseteq f^{-1}(mC(B, r))$ for $B \in I^Y$.

6 $f^{-1}(mI(B, r)) \subseteq m\alpha I(f^{-1}(B), r)$ for $B \in I^Y$.

Definition 2.5.4 [7] Let $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ be a mapping on two r-FMS's. Then

1 f is said to be *fuzzy r-M α-open* if for fuzzy r-minimal open set A in X, $f(A)$ is fuzzy r-minimal α-open in Y;

2 f is said to be *fuzzy r-M α-closed* if for fuzzy r-minimal closed set A in X, $f(A)$ is fuzzy r-minimal α-closed in Y.

Theorem 2.5.5 [7] Let $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ be a mapping on two r-FMS's. Then the following statements are equivalent:

1 f is fuzzy r-M α-open.

2 $f(mI(A, r)) \subseteq m\alpha I(f(A), r)$ for all $A \in I^X$.

3 $mI(f^{-1}(B), r) \subseteq f^{-1}(m\alpha I(B, r))$ for all $B \in I^Y$.

Theorem 2.5.6 [7] Let $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ be a mapping on two r-FMS's. Then the following statements are equivalent:

1 f is fuzzy r-M α-closed.

2 $m\alpha C(f(A), r) \subseteq f(mC(A, r))$ for all $A \in I^X$.

3 $f^{-1}(m\alpha C(B, r)) \subseteq mC(f^{-1}(B), r)$ for all $B \in I^Y$.

Definition 2.5.7 [7] Let $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ be a mapping on two r-FMS's. Then

1 f is said to be *fuzzy r-$M^*\alpha$-open* if for fuzzy r-minimal open set A in X, $f(A)$ is fuzzy r-minimal open in Y;

2 f is said to be *fuzzy r-$M^*\alpha$-closed* if for fuzzy r-minimal closed set A in X, $f(A)$ is fuzzy r-minimal closed in Y.

Theorem 2.5.8 [7] Let $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ be a mapping on two r-FMS's. If

1 f is fuzzy r-$M^*\alpha$-open.

2 $f(m\alpha I(A, r)) \subseteq mI(f(A), r)$ for all $A \in I^X$.

3 $m\alpha I(f^{-1}(B), r) \subseteq f^{-1}(mI(B, r))$ for all $B \in I^Y$.

Then $1 \Rightarrow 2 \Leftrightarrow 3$.

Theorem 2.5.9 [7] Let $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ be a mapping on two r-FMS's. If

1 f is fuzzy r-M^* α-closed.

2 $mC(f(A), r) \subseteq f(m\alpha C(A, r))$ for all $A \in I^X$.

3 $f^{-1}(mC(B), r)) \subseteq m\alpha C(f^{-1}(B), r)$ for all $B \in I^Y$.

Then $1 \Rightarrow 2 \Leftrightarrow 3$.

Corollary 2.5.10 [7] Let $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ be a mapping on two r-FMS's. If $\mathcal{N}$ has the property $(\mathcal{U})$, then the following statements are equivalent:

1 f is fuzzy r-$M^*\alpha$-open.

2 $f(m\alpha I(A, r)) \subseteq mI(f(A), r)$ for all $A \in I^X$.

3 $m\alpha I(f^{-1}(B), r) \subseteq f^{-1}(mI(B, r))$ for all $B \in I^Y$.

Corollary 2.5.11 [7] Let $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ be a mapping on two r-FMS's. If $\mathcal{N}$ has the property $(\mathcal{U})$, then the following statements are equivalent:

1 f is fuzzy r-$M^*\alpha$-closed.

2 $mC(f(A), r) \subseteq f(m\alpha C(A, r)$ for all $A \in I^X$.

3 $f^{-1}(mC(B), r)) \subseteq m\alpha C(f^{-1}(B), r)$ for all $B \in I^Y$.

CHAPTER 3

ORDINARY SMOOTH r-MINIMAL STRUCTURE SPACES

In this chapter, we define the ordinary smooth r-minimal structure spaces which study the concepts of open set, closed set, closure and interior it intersects on such. The study properties of opens mapping, continuous mapping and compactness. And introduced many relationships between some types of generalized closed sets and r-mb generalized closed sets, Also we study characterization of extremely disconnected and T_{gs} spaces on ordinary smooth r-minimal spaces.

3.1 Ordinary Smooth r-Minimal Compactness

First, we define the open set, closed set, closure, interior, continuous mapping and opens mapping in ordinary smooth r-minimal structure spaces and some basic properties. For each a nonempty set X, let 2^X the set of all subsets of a set X.

Definition 3.1.1 Let X be a nonempty set and $r \in (0, 1]$. A mapping $\mathcal{M} : 2^X \to I$ is said to have an ordinary smooth r-minimal structure if the family

$$\mathcal{M}_r = \{A \in 2^X : \mathcal{M}(A) \geq r\}$$

contains $\emptyset$ and X.

Then the $(X, \mathcal{M})$ is called an *ordinary smooth r-minimal structure space* (simply, r-OSMS). Every member of $\mathcal{M}_r$ is called an *ordinary smooth r-minimal open set* (simply, r-OSM open set). A subset A of X is called an *ordinary smooth r-minimal closed set* (simply, r-OSM closed set) if the complement of A (simply, A^C) is an ordinary smooth r-minimal open set.

Example 3.1.2 Let $X = \{a, b, c, d\}$ and $\mathcal{M} : 2^X \to I$.
Let $A \in 2^X$. Then

$$\mathcal{M}(A) = \begin{cases} 0.9, & \text{if } A = X, A = \emptyset, \\ 0.8, & \text{if } A = \{c\}, \{c, d\}, \{d\}, \\ 0.7, & \text{if } A = \{b\}, \{b, d\}, \{b, c\}, \{b, c, d\}, \\ 0.4, & \text{if } A = \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \\ 0.2, & \text{if } A = \{a\}, \{a, d\}, \{a, b\}, \{a, c\}. \end{cases}$$

Let $r = \frac{1}{4}$, we get that $\mathcal{M}_{\frac{1}{4}} = \{A \in 2^X : \mathcal{M}(A) \geq \frac{1}{4}\}$.
$\mathcal{M}_{\frac{1}{4}} = \{\{b\}, \{b, d\}, \{b, c\}, \{b, c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{c\}, \{c, d\}, \{d\}, \emptyset, X\}$.
Thus $(X, \mathcal{M})$ is $\frac{1}{4}$-OSMS. Then $\{b\}, \{b, d\}, \{b, c\}, \{b, c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}$, $\{c\}, \{c, d\}, \{d\}, \emptyset, X$ are $\frac{1}{4}$-OSM open sets and $\{a, c, d\}, \{a, c\}, \{a, d\}, \{a\}, \{d\}, \{c\}, \{b\}$, $\{a, b, d\}, \{a, b\}, \{a, b, c\}, \emptyset, X$ are $\frac{1}{4}$-OSM closed sets.

Definition 3.1.3 Let $(X, \mathcal{M})$ be an r-OSMS and $r \in (0, 1]$. The r-OSM closure and the r-OSM interior of A, denote by $mC(A, r)$ and $mI(A, r)$, respectively, are define as

$$mC(A, r) = \cap\{B \in 2^X : B^C \in \mathcal{M}_r \text{ and } A \subseteq B\},$$

$$mI(A, r) = \cup\{B \in 2^X : B \in \mathcal{M}_r \text{ and } B \subseteq A\}.$$

Example 3.1.4 From Example 3.1.2.
Let $r = \frac{1}{4}$ and $\mathcal{M}_{\frac{1}{4}}$. Let $A = \{b, c\}$,

$$\begin{aligned} mC(\{b, c\}, \tfrac{1}{4}) &= \cap\{B \in 2^X : B^C \in \mathcal{M}_{\frac{1}{4}} \text{ and } \{b, c\} \subseteq B\} \\ &= \cap\{\{a, b, c\}, X\} = \{a, b, c\}. \end{aligned}$$

Let $A = \{a, c\}$,

$$\begin{aligned} mI(\{a, c\}, \tfrac{1}{4}) &= \cup\{B \in 2^X : B \in \mathcal{M}_{\frac{1}{4}} \text{ and } B \subseteq \{a, c\}\} \\ &= \cup\{\{c\}, \emptyset\} = \{c\}. \end{aligned}$$

Theorem 3.1.5 Let $(X, \mathcal{M})$ be an r-OSMS and $A, B \in 2^X$.

1 $mI(A, r) \subseteq A$.

2 If A is r-OSM open, then $mI(A, r) = A$.

3 $A \subseteq mC(A, r)$.

4 If A is r-OSM closed, then $mC(A, r) = A$.

5 If $A \subseteq B$, then $mI(A, r) \subseteq mI(B, r)$ and $mC(A, r) \subseteq mC(B, r)$.

6 $mI(A, r) \cap mI(B, r) \supseteq mI(A \cap B, r)$ and $mC(A, r) \cup mC(B, r) \subseteq mC(A \cup B, r)$.

7 $mI(mI(A, r), r) = mI(A, r)$ and $mC(mC(A, r), r) = mC(A, r)$.

8 $X - mC(A, r) = mI(X - A, r)$ and $X - mI(A, r) = mC(X - A, r)$.

Proof. (1) Let $x \in mI(A, r)$. There exists $B \in \mathcal{M}_r$, such that $B \subseteq A$ and $x \in B$. Thus $x \in A$. Therefore $mI(A, r) \subseteq A$.

(2) Let A be r-OSM open. Since $A \subseteq A$ and $A \in \mathcal{M}_r$, then $A \subseteq mI(A, r)$. This implies that $mI(A, r) = A$.

(3) Let $x \in A$, then $x \in \cap\{B \in 2^X : B^C \in \mathcal{M}_r \text{ and } A \subseteq B\} = mC(A, r)$. Thus $A \subseteq mC(A, r)$.

(4) Let A be r-OSM closed. Since $A \subseteq A$ and $A^C \in \mathcal{M}_r$, we have $mC(A, r) \subseteq A$. Thus $mC(A, r) = A$.

(5) Assume $A \subseteq B$, we have to show that $mI(A, r) \subseteq mI(B, r)$.
Let $A \subseteq B$ and let $x \in mI(A, r)$, there exists $U \in \mathcal{M}_r$ such that $x \in U$ which $U \subseteq A$. Since $A \subseteq B$, we have $U \subseteq B$. Thus $x \in \cup\{U' \in 2^X : U \subseteq B \text{ and } U' \in \mathcal{M}_r\} = mI(B, r)$. Hence $mI(A, r) \subseteq mI(B, r)$.
Now to show that suppose $x \notin mC(B, r)$, there exists $F^C \in \mathcal{M}_r$ which $B \subseteq F$ but $x \notin F$. Since $A \subseteq B$, we have $A \subseteq F$. Hence $x \notin mC(A, r)$.
Therefore $mC(A, r) \subseteq mC(B, r)$.

(6) Since $A \cap B \subseteq A$, $A \cap B \subseteq B$ and by (5),
we have $mI(A \cap B, r) \subseteq mI(A, r)$ and $mI(A \cap B, r) \subseteq mI(B, r)$.
So $mI(A \cap B, r) \subseteq mI(A, r) \cap mI(B, r)$.
And since $A \subseteq A \cup B$, $B \subseteq A \cup B$ and by (5),

we get that $mC(A,r) \subseteq mC(A\cup B,r)$ and $mC(B,r) \subseteq mC(A\cup B,r)$.
Thus $mC(A,r)\cup mC(B,r) \subseteq mC(A\cup B,r)$.

(7) Let $x \in X - mC(A,r)$ Then $x \notin mC(A,r)$, there exists $F^C \in \mathcal{M}_r$ which $A \subseteq F$ but $x \notin F$. So $x \in X - F$ such that $X - F \subseteq X - A$. Then $x \in \cup\{F^C \in 2^X : X - F \subseteq X - A$ and $F^C \in \mathcal{M}_r\}$. Thus $x \in mI(X - A, r)$. Therefore $X - mC(A,r) \subseteq mI(X - A, r)$. Now to show that let $x \in mI(X - A, r)$, there exists $F \in \mathcal{M}_r$ such that $x \in F$ which $F \subseteq X - A$. Thus $x \notin X - F$ for all $F^C \in \mathcal{M}_r$. Then $x \notin \cap\{F^C \in 2^X : A \subseteq X - F$ and $F^C \in \mathcal{M}_r\}$. Hence $x \notin mC(A,r)$. Therefore $x \in X - mC(A,r)$. This implies that $X - mC(A,r) = mI(X - A, r)$.
Now to show that $X - mI(A,r) = mC(X - A, r)$. We have

$$\begin{aligned} X - mI(A,r) &= X - mI(X - (X - A), r) \\ &= X - (X - mC(X - A, r) \\ &= mC(X - A, r). \end{aligned}$$

Hence $X - mI(A,r) = mC(X - A, r)$.

(8) Let $x \in mI(A,r)$. Since $mI(A,r) \subseteq \{B : B \subseteq mI(A,r), B \in \mathcal{M}_r\}$.
Thus $x \in \cup\{B : B \subseteq mI(A,r), B \in \mathcal{M}_r\}$. So $x \in mI(mI(A,r),r)$.
Hence $mI(A,r) \subseteq mI(mI(A,r),r)$. And since by (1), $mI(A,r) \subseteq A$.
By (5), $mI(mI(A,r),r) \subseteq mI(A,r)$. Therefore $mI(mI(A,r) = mI(A,r)$.
Now to show that $mC(mC(A,r) = mC(A,r)$. Consider

$$\begin{aligned} mC(mC(A,r),r) &= mC(mC(X - (X - A), r), r) \\ &= mC(X - mI(X - A, r)), r) \\ &= X - mI(X - A, r) \\ &= mC(A,r). \end{aligned}$$

Hence $mC(mC(A,r) = mC(A,r)$. □

Definition 3.1.6 Let $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ be a mapping on two r-OSMS's. Then f is said to be

1 *ordinary smooth r-M continuous mapping* (simply, r-M continuous) if for every $A \in \mathcal{N}_r$, $f^{-1}(A)$ is in $\mathcal{M}_r$.

2 *ordinary smooth* r-M *open mapping* (simply, r-M open) if for every $A \in \mathcal{M}_r$, $f(A)$ is in $\mathcal{N}_r$.

Example 3.1.7 Let $X = \{a,b,c,d\}$ and $\mathcal{M} : 2^X \to I$. Define

$$\mathcal{M}(A) = \begin{cases} 0.9, & \text{if } A = X, A = \emptyset, \\ 0.7, & \text{if } A = \{c\}, \{c,d\}, \{d\}, \\ 0.6, & \text{if } A = \{b\}, \{b,d\}, \{b,c\}, \{b,c,d\}, \\ 0.4, & \text{if } A = \{a,b,c\}, \{a,b,d\}, \{a,c,d\}, \\ 0.2, & \text{if } A = \{a\}, \{a,d\}, \{a,b\}, \{a,c\}. \end{cases}$$

Let $Y = \{x,y,z\}$ and $\mathcal{N} : 2^Y \to I$. Define

$$\mathcal{N}(A) = \begin{cases} 1, & \text{if } A = Y, A = \emptyset, \\ \frac{1}{2}, & \text{if } A = \{z\}, \\ \frac{1}{3}, & \text{if } A = \{y\}, \{y,z\}, \\ \frac{1}{4}, & \text{if } A = \{x\}, \{x,y\}, \{x,z\}. \end{cases}$$

Let $r = \frac{1}{2}$, we have

$\mathcal{N}_{\frac{1}{2}} = \{\emptyset, Y, \{z\}\}$ and $\mathcal{M}_{\frac{1}{2}} = \{\emptyset, X, \{b\}, \{b,d\}, \{b,c\}, \{b,c,d\}, \{c\}, \{c,d\}, \{d\}\}$.

Define $f : X \to Y$, by $f(a) = x$, $f(b) = y$, $f(c) = f(d) = z$.

From Definition 3.1.6 (1), then f is $\frac{1}{2}$-M continuous. But f is not $\frac{1}{2}$-M open.

Let $r = \frac{1}{4}$, we have

$\mathcal{M}_{\frac{1}{4}} = \{\emptyset, X, \{b\}, \{b,d\}, \{b,c\}, \{b,c,d\}, \{c\}, \{c,d\}, \{d\}, \{a,b,c\}, \{a,b,d\}, \{a,c,d\}\}$ and $\mathcal{N}_{\frac{1}{4}} = \{\emptyset, Y, \{x\}, \{x,y\}, \{x,z\}, \{y\}, \{y,z\}, \{z\}\}$.

Define $f : X \to Y$, by $f(a) = x$, $f(b) = y$, $f(c) = f(d) = z$.

From Definition 3.1.6 (2), then f is $\frac{1}{4}$-M open. But f is not $\frac{1}{4}$-M continuous.

Later, we will define the concepts of r-OSM compact, r-OSM almost compact and r-OSM nearly compact on r-OSMS and investigate the relationships between r-M continuous and such types of r-OSM compact.

Definition 3.1.8 Let $(X, \mathcal{M})$ be an r-OSMS and $\{A_i \in 2^X : i \in J\}$. A is called an *ordinary smooth r-minimal cover* (simply, r-OSM cover) of X if $\bigcup_{i\in J} A_i = X$. It is an *ordinary smooth r-minimal open cover* (simply, r-OSM open cover) if each A_i is an r-OSM set. $\{B_i \in 2^X : i \in J\}$ is called an *ordinary smooth r-minimal open cover of* $B \subseteq X$ if $B \subseteq \cup\{B_i \in 2^X : i \in J\}$.

Definition 3.1.9 Let $(X, \mathcal{M})$ be an r-OSMS. A subset A of X is said to be an *ordinary smooth r-minimal compact* (simply, r-OSM compact) if every r-OSM open cover $\{A_i \in \mathcal{M}_r : i \in J\}$ of A has a finite subcover.

Theorem 3.1.10 Let $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ be an r-M continuous mapping on two r-OSMS's. If A is an r-OSM compact set, then $f(A)$ is also an r-OSM compact set.

Proof. Let A be r-OSM compact and $\{B_i \in 2^Y : i \in J\}$ be r-OSM open cover of $f(A)$ in Y, then $\{f^{-1}(B_i) : i \in J\}$ is r-OSM open cover of A in X.
Since A is an r-OSM compact set, there exists $J_0 = \{j_1, j_2, ..., j_n\} \subseteq J$ such that $A \subseteq \bigcup_{i\in J_0} f^{-1}(B_i)$, thus $f(A) \subseteq f(\bigcup_{i\in J_0} f^{-1}(B_i)) = \bigcup_{i\in J_0} f(f^{-1}(B_i)) \subseteq \bigcup_{i\in J_0} B_i$.
Hence $f(A)$ is an r-OSM compact set. □

Definition 3.1.11 Let $(X, \mathcal{M})$ be an r-OSMS. A subset A in X is said to be an *ordinary smooth r-minimal almost compact* (simply, r-OSM almost compact) if for every r-OSM open cover $\{A_i \in 2^X : i \in J\}$ of A, there exists $J_0 = \{j_1, j_2, ..., j_n\} \subseteq J$ such that $A \subseteq \bigcup_{i\in J_0} mC(A_i, r)$.

Theorem 3.1.12 Let $(X, \mathcal{M})$ be an r-OSMS. If a subset A in X is r-OSM compact, then it is also r-OSM almost compact.

Proof. Let A be r-OSM compact and $\{B_i \in 2^X : i \in J\}$ be r-OSM open cover of A. Since A is r-OSM compact, there exists $J_0 = \{j_1, j_2, ..., j_n\} \subseteq J$ such that $A \subseteq \bigcup_{i\in J_0} B_i$. By Theorem 3.1.5 (3), we have $\bigcup_{i\in J_0} B_i \subseteq \bigcup_{i\in J_0} mC(B_i, r)$. Hence $A \subseteq \bigcup_{i\in J_0} mC(B_i, r)$. □

Example 3.1.13 Let $X = (0, 1)$ and $n \in \mathbb{N}$.

Let $A_n = (0, \frac{1}{n})$ and $B_n = (\frac{1}{n}, \frac{n-1}{n})$ which $n = 3, 4, \ldots$

$$\mathcal{M}(A) = \begin{cases} 1, & \text{if } A = X, A = \emptyset, \\ 0.8, & \text{if } A = (0, \frac{1}{n}); n = 3, 4, \ldots \\ 0.6, & \text{if } A = (\frac{1}{n}, \frac{n-1}{n}); n = 3, 4, \ldots \\ 0, & \text{if otherwise} \end{cases}$$

Let $A = \{A_n \in 2^X : n \in \mathbb{N}\}$ is $\frac{1}{2}$-OSM open cover, then there exists $J_0 = \{A_3, B_3\} \subseteq \mathbb{N}$ such that $A \subseteq \bigcup_{n \in J_0} \{mC(A_3, \dfrac{1}{2}), mC(B_3, \dfrac{1}{2})\} = \bigcup_{n \in J_0} \{(0, \dfrac{1}{3}), [\dfrac{2}{3}, 1], [\dfrac{1}{3}, 1]\} = (0, 1) = X$. Thus X is $\frac{1}{2}$-OSM almost compact. Since $\{A_n : n \in \mathbb{N}\}$ is $\frac{1}{2}$-OSM open cover, then not finite subcover of A, and so X not is $\frac{1}{2}$-OSM compact.

Definition 3.1.14 Let X be a nonempty set and $\mathcal{M} : 2^X \to I$ a family on X. The family $\mathcal{M}$ has the property $(\mathcal{U})$ if for $A_i \in \mathcal{M}_r (i \in J)$,

$$\mathcal{M}(\cup A_i) \geq \wedge \mathcal{M}(A_i).$$

Theorem 3.1.15 Let $(X, \mathcal{M})$ be an r-OSMS and $\mathcal{M}$ has the property $(\mathcal{U})$. Then

1 $mI(A, r) = A$ if and only if $A \in \mathcal{M}_r$ for $A \in 2^X$.

2 $mC(A, r) = A$ if and only if $A^C \in \mathcal{M}_r$ for $A \in 2^X$.

Proof. (1) Let $A \in 2^X$ be such that $mI(A, r) = A$. By $\mathcal{M}$ has the property $(\mathcal{U})$, we have$A \in \mathcal{M}_r$. Conversely, let $A \in \mathcal{M}_r$, then A is an r-OSM open set. By Theorem 3.1.5 (2), $mI(A, r) = A$.

(2) Let $A \in 2^X$ be such that $mC(A, r) = A$. By $\mathcal{M}$ has the property $(\mathcal{U})$, we have $A^C \in \mathcal{M}_r$. Conversely, let $A^C \in \mathcal{M}_r$, then A is an r-OSM closed set. By Theorem 3.1.5 (4), $mC(A, r) = A$. □

Theorem 3.1.16 Let $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ be a mapping on two r-OSMS's. Then the following statements are equivalent:

1 f is r-M continuous.

2 $f^{-1}(B)$ is an r-OSM closed set, for each r-OSM closed set B in Y.

Proof. (1) ⇒ (2) Let B be an r-OSM closed set. Then $Y-B$ is an r-OSM open set. Since f is an r-M continuous, $X - f^{-1}(B) = f^{-1}(Y-B)$ is an r-OSM open set. Therefore $f^{-1}(B)$ is an r-OSM closed set.

(2) ⇒ (1) Let $B \in 2^Y$ and let B be an r-OSM closed set. Then $Y-B$ is an r-OSM open set. Then $f^{-1}(Y-B)$ is an r-OSM open set in X. Thus f is r-M continuous. □

Theorem 3.1.17 Let $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ be a mapping on two r-OSMS's. Then the following statements are hold:

1 If f is r-M continuous, then $f(mC(A,r)) \subseteq mC(f(A),r)$ for all $A \in 2^X$.

2 If $f^{-1}(mI(B,r)) \subseteq mI(f^{-1}(B),r)$, for all $B \in 2^Y$ is true and $\mathcal{M}$ has the property $(\mathcal{U})$, then f is r-M continuous.

Proof. (1) Let f be r-M continuous and let $A \in 2^X$, then $f^{-1}(A) \in 2^X$. Consider

$$\begin{aligned} f^{-1}(mC(f(A),r) &= f^{-1}(\cap\{F \in 2^X : f(A) \subseteq F, F^C \in \mathcal{M}_r\}) \\ &= \cap\{f^{-1}(F) \in 2^X : A \subseteq f^{-1}(f(A)) \subseteq f^{-1}(F), F^C \in \mathcal{M}_r\} \\ &\supseteq \cap\{K \in 2^X : A \subseteq K, K^C \in \mathcal{M}_r\} \\ &= mC(A,r). \end{aligned}$$

Then $mC(A,r) \subseteq f^{-1}(mC(f(A),r)$. Thus $f(mC(A,r)) \subseteq f(f^{-1}(mC(f(A),r)))$ $\subseteq mC(f(A),r)$. Hence $f(mC(A,r)) \subseteq mC(f(A),r)$.

(2) Let B be r-OSM open in Y. Then $f^{-1}(B) = f^{-1}(mI(B,r)) \subseteq mI(f^{-1}(B),r)$. Thus $f^{-1}(B) \subseteq mI(f^{-1}(B),r) \subseteq f^{-1}(B)$. This implies that $f^{-1}(B) = mI(f^{-1}(B),r)$. Since $\mathcal{M}$ has the property $(\mathcal{U})$ and Theorem 3.1.15 (1), $f^{-1}(B)$ is r-OSM open in X. Hence f is r-M continuous. □

Theorem 3.1.18 Let $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ be a mapping on two r-OSMS's. Then the following statements are equivalent:

1 $f(mC(A,r)) \subseteq mC(f(A),r)$ for $A \in 2^X$.

2 $mC(f^{-1}(B),r) \subseteq f^{-1}(mC(B,r))$ for $B \in 2^Y$.

3 $f^{-1}(mI(B,r)) \subseteq mI(f^{-1}(B),r)$ for $B \in 2^Y$.

Proof. (1) ⇒ (2) Let $B \in 2^Y$, then $f^{-1}(B) \in 2^X$.
By assumption, then $f(mC(f^{-1}(B), r)) \subseteq mC(f(f^{-1}(B)), r)$.
But $mC(f(f^{-1}(B)), r) \subseteq mC(B, r)$. So $f(mC(f^{-1}(B), r)) \subseteq mC(B, r)$.
Thus $mC(f^{-1}(B), r) \subseteq f^{-1}(f(mC(f^{-1}(B), r))) \subseteq f^{-1}(mC(B, r))$.
Hence $mC(f^{-1}(B), r) \subseteq f^{-1}(mC(B, r))$.
(2) ⇒ (3) For $B \in 2^Y$. Since $mI(B, r) = Y - mC(Y - B, r)$

$$
\begin{aligned}
f^{-1}(mI(B, r)) &= f^{-1}(Y - mC(Y - B, r)) \\
&= f^{-1}(Y) - f^{-1}(mC(Y - B, r)) \\
&= X - f^{-1}(mC(Y - B, r)) \\
&\subseteq X - mC(f^{-1}(Y - B), r) \\
&= mI(f^{-1}(B), r).
\end{aligned}
$$

Hence $f^{-1}(mI(B, r)) \subseteq mI(f^{-1}(B), r)$.
(3) ⇒ (1) For $A \in 2^X$. Consider,

$$
\begin{aligned}
mC(A, r) &\subseteq mC(f^{-1}(f(A)), r) \\
&\subseteq mC(f^{-1}(Y - f(X - A)), r) \\
&= mC(X - f^{-1}(f(X - A)), r) \\
&= X - mI(f^{-1}(f(X - A)), r) \\
&\subseteq X - f^{-1}(mI(f(X - A), r)) \\
&= f^{-1}(Y - mI(f(X - A), r)) \\
&= f^{-1}(mC(Y - f(X - A), r)) \\
&= f^{-1}(mC(f(A), r)).
\end{aligned}
$$

Thus $f(mC(A, r)) \subseteq f(f^{-1}(mC(f(A), r)) \subseteq mC(f(A), r)$.
Hence $f(mC(A, r)) \subseteq mC(f(A), r)$. □

Theorem 3.1.19 Let $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ be an r-M continuous mapping on two r-OSMS's. If A is an r-OSM almost compact set, then $f(A)$ is also an r-OSM almost compact set.

Proof. Let A is an r-OSM almost compact set and $\{B_i \in 2^Y : i \in J\}$ be r-OSM open cover of $f(A)$ in Y. Then $\{f^{-1}(B_i) \in 2^X : i \in J\}$ is r-OSM open cover of A

in X. By A is an r-OSM almost compact set, there exists $J_0 = \{j_1, j_2, ..., j_n\} \subseteq J$, such that $A \subseteq \bigcup_{i \in J_0} mC(f^{-1}(B_i), r)$. From Theorem 3.1.17 and Theorem 3.1.18 (2), we have $\bigcup_{i \in J_0} mC(f^{-1}(B_i), r) \subseteq \bigcup_{i \in J_0} f^{-1}(mC(B_i, r)) = f^{-1}(\bigcup_{i \in J_0} mC(B_i, r))$. And so $A \subseteq f^{-1}(\bigcup_{i \in J_0} mC(B_i, r))$. Thus $f(A) \subseteq \bigcup_{i \in J_0} mC(B_i, r)$. Hence $f(A)$ is an r-OSM almost compact set. □

Definition 3.1.20 Let $(X, \mathcal{M})$ be an r-OSMS. A subset A in X is said to be an *ordinary smooth r-minimal nearly compact* (simply, r-OSM nearly compact) if for every r-OSM open cover $\{A_i : i \in J\}$ of A, there exists $J_0 = \{j_1, j_2, ..., j_n\} \subseteq J$ such that $A \subseteq \bigcup_{i \in J_0} mI(mC(A_i, r), r)$.

Theorem 3.1.21 Let $(X, \mathcal{M})$ be an r-OSMS. If a subset A in X is an r-OSM compact, then it is an r-OSM nearly compact.

Proof. Let A be r-OSM compact and $\{B_i \in 2^X : i \in J\}$ be an r-OSM open cover of A. Since A is an r-OSM compact, there exists $J_0 = \{j_1, j_2, ..., j_n\} \subseteq J$ such that $A \subseteq \bigcup_{i \in J_0} B_i$. By Theorem 3.1.5 (3), we have $\bigcup_{i \in J_0} B_i \subseteq \bigcup_{i \in J_0} mC(B_i, r)$. and by Theorem 3.1.5 (5), $\bigcup_{i \in J_0} B_i = \bigcup_{i \in J_0} mI(B_i, r) \subseteq \bigcup_{i \in J_0} mI(mC(B_i, r), r)$. Hence $A \subseteq \bigcup_{i \in J_0} mI(mC(B_i, r), r)$. Hence A is an r-OSM nearly compact. □

Theorem 3.1.22 Let $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ be a mapping on two r-OSMS's. Then the following statements are equivalent:

1 $f(mI(A, r)) \subseteq mI(f(A), r)$ for $A \in 2^X$.

2 $mI(f^{-1}(B), r) \subseteq f^{-1}(mI(B, r))$ for $B \in 2^Y$.

Proof. (1) ⇒ (2) For $B \in 2^Y$. It follow from (2), we have $f^{-1}(B) \in 2^X$. Since $f(mI(f^{-1}(B), r)) \subseteq mI(f(f^{-1}(B), r)) \subseteq mI(B, r)$. Thus $f(mI(f^{-1}(B), r)) \subseteq mI(B, r)$. Hence $mI(f^{-1}(B), r) \subseteq f^{-1}(f(mI(f^{-1}(B), r))) \subseteq f^{-1}(mI(B, r))$.
Therefore $mI(f^{-1}(B), r) \subseteq f^{-1}(mI(B, r))$.
(2) ⇒ (1) For $A \in 2^X$, we have $f(A) \in 2^Y$. Since $mI(A, r) \subseteq mI(f^{-1}(f(A)), r) \subseteq f^{-1}(mI(f(A), r))$. Hence $f(mI(A, r)) \subseteq f(f^{-1}(mI(f(A), r))) \subseteq mI(f(A), r)$.
Therefore $f(mI(A, r)) \subseteq mI(f(A), r)$. □

Theorem 3.1.23 Let $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ be a mapping on two r-OSMS's. Then the following statements are equivalent:

1 If f is r-M open, then $f(mI(A, r)) \subseteq mI(f(A), r)$ for $A \in 2^X$.

2 If $mI(f^{-1}(B), r) \subseteq f^{-1}(mI(B, r))$ for $B \in 2^Y$ and $\mathcal{N}$ has the property ($\mathcal{U}$), then f is r-M open.

Proof. (1) Let f be r-M open and let $A \in 2^X$. Consider,

$$\begin{aligned} f(mI(A, r)) &= f(\cup\{G \in 2^X : G \subseteq A \text{ and } G \in \mathcal{M}_r\}) \\ &= \cup\{f(G) \in 2^Y : f(G) \subseteq f(A) \text{ and } f(G) \in \mathcal{M}_r\}) \\ &\subseteq \cup\{U \in 2^X : U \subseteq f(A) \text{ and } U \in \mathcal{M}_r\}) \\ &= mI(f(A), r). \end{aligned}$$

Hence $f(mI(A, r)) \subseteq mI(f(A), r)$.

(2) Let B be r-OSM open in X. Then

$$\begin{aligned} f(B) &= f(mI(B, r)) \\ &\subseteq f(mI(f^{-1}(f(B)), r)) \\ &\subseteq f(f^{-1}(mI(f(B), r))) \\ &\subseteq mI(f(B), r) \end{aligned}$$

Thus $mI(f(B), r) = f(B)$. By $\mathcal{N}$ has the property ($\mathcal{U}$) and Theorem 3.1.15 (1), f is r-M open. □

Theorem 3.1.24 Let $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ be an r-M continuous and r-M open mapping on two r-OSMS's. If A is an r-OSM nearly compact set, then $f(A)$ is an r-OSM nearly compact set.

Proof. Let A is an r-OSM nearly compact set and $\{B_i \in 2^X : i \in J\}$ be r-OSM open cover of $f(A)$ in Y, then $\{f^{-1}(B_i) : i \in J\}$ is r-OSM open cover of A in X.
By r-OSM nearly compact set, there exists $J_0 = \{j_1, j_2, ..., j_n\} \subseteq J$ such that $A \subseteq \bigcup_{i \in J_0} mI(mC(f^{-1}(B_i), r), r)$. From Theorem 3.1.18 and Theorem 3.1.23 (1), it that

follows

$$
\begin{aligned}
f(A) &\subseteq \bigcup_{i \in J_0} f(mI(mC(f^{-1}(B_i), r), r)) \\
&\subseteq \bigcup_{i \in J_0} mI(f(mC(f^{-1}(B_i), r)), r) \\
&\subseteq \bigcup_{i \in J_0} mI(mC(f(f^{-1}(B_i)), r), r) \\
&\subseteq \bigcup_{i \in J_0} mI(mC(B_i, r), r),
\end{aligned}
$$

and so $f(A) \subseteq \bigcup_{i \in J_0} mI(mC(B_i, r), r)$. Hence $f(A)$ is an r-OSM nearly compact set. □

Definition 3.1.25 Let $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ be a mapping on two r-OSMS's. Then f is said to be an *ordinary smooth weakly r-M continuous* (simply, r-M weak continuous) if for $x \in X$ and each r-OSM open set V containing $f(x)$, then there is an r-OSM open set U containing x such that $f(U) \subseteq mC(V, r)$.

Example 3.1.26 Let $X = \{a, b, c\}$ and $\mathcal{M} : 2^X \to I$. Define

$$
\mathcal{M}(A) = \begin{cases} 0.8, & \text{if } A = X, A = \emptyset, \\ 0.6, & \text{if } A = \{c\}, \{b\}, \{b, c\}, \\ 0.4, & \text{if } A = \{a, b\}, \{a, c\}, \\ 0.2, & \text{if } A = \{a\}. \end{cases}
$$

Let $Y = \{x, y, z\}$ and $\mathcal{N} : 2^Y \to I$. Define

$$
\mathcal{N}(A) = \begin{cases} 1, & \text{if } A = Y, A = \emptyset, \\ 0.7, & \text{if } A = \{y\}, \{z\}, \{y, z\} \\ 0.4, & \text{if } A = \{x, y\}, \{x, z\}, \\ 0.1, & \text{if } A = \{x\}. \end{cases}
$$

Let $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ and $r = \frac{1}{4}$ define as follows : $f(a) = x, f(b) = y, f(c) = z$. Then $\mathcal{M}_{\frac{1}{4}} = \{\{a, b\}, \{a, c\}, \{c\}, \{b\}, \{b, c\}, \emptyset, X\}$, $\mathcal{N}_{\frac{1}{4}} = \{\{x, y\}, \{x, z\}, \{y\}, \{z\}, \{y, z\}, \emptyset, Y\}$. Consider $a \in X$, $f(a) = x$ is a member of $\frac{1}{4}$-OSM open set $\{x, z\}, \{x, y\}, Y$.

- If $V = \{x, y\}$ then, we choose $U = \{b\} \in \mathcal{M}_{\frac{1}{4}}$ such that $f(U) = f(\{b\}) = \{y\} \subseteq \{x, y\} = mC(\{x, y\}, \frac{1}{4})$.
- If $V = \{x, z\}$ then , we choose $U = \{a, c\} \in \mathcal{M}_{\frac{1}{4}}$ such that $f(U) = f(\{a, c\}) = \{x, z\} \subseteq \{x, z\} = mC(\{x, z\}, \frac{1}{4})$.
- If $V = Y$ then, we choose $U = \{a, c\} \in \mathcal{M}_{\frac{1}{4}}$ such that $f(U) = f(\{a, c\}) = \{y, z\} \subseteq Y = mC(Y, \frac{1}{4})$.

Consider $b \in X$, $f(b) = y$ is a member of $\frac{1}{4}$-OSM open set $\{y\}, \{x, y\}, \{y, z\}, Y$.

- If $V = \{y\}$ then, we choose $U = \{a, b\} \in \mathcal{M}_{\frac{1}{4}}$ such that $f(U) = f(\{a, b\}) = \{y\} \subseteq Y = mC(\{y\}, \frac{1}{4})$.
- If $V = \{x, y\}$ then, we choose $U = \{b\} \in \mathcal{M}_{\frac{1}{4}}$ such that $f(U) = f(\{b\}) = \{y\} \subseteq \{x, y\} = mC(\{x, y\}, \frac{1}{4})$.
- If $V = \{y, z\}$ then, we choose $U = \{b, c\} \in \mathcal{M}_{\frac{1}{4}}$ such that $f(U) = f(\{b, c\}) = \{y, z\} \subseteq Y = mC(\{y, z\}, \frac{1}{4})$.
- If $V = Y$ then, we choose $U = \{a, c\} \in \mathcal{M}_{\frac{1}{4}}$ such that $f(U) = f(\{a, c\}) = \{y, z\} \subseteq Y = mC(Y, \frac{1}{4})$.

Consider $c \in X$, $f(c) = z$ is a member of $\frac{1}{4}$-OSM open set $\{x, z\}, \{y, z\}, \{z\}, Y$.

- If $V = \{x, z\}$ then, we choose $U = \{c\} \in \mathcal{M}_{\frac{1}{4}}$ such that $f(U) = f(\{c\}) = \{z\} \subseteq \{x, z\} = mC(\{\{x, z\}, \frac{1}{4})$.
- If $V = \{y, z\}$ then, we choose $U = \{b\} \in \mathcal{M}_{\frac{1}{4}}$ such that $f(U) = f(\{b\}) = \{y\} \subseteq Y = mC(\{y, z\}, \frac{1}{4})$.
- If $V = \{z\}$ then, we choose $U = \{c\} \in \mathcal{M}_{\frac{1}{4}}$ such that $f(U) = f(\{c\}) = \{z\} \subseteq \{z\} = mC(\{z\}, \frac{1}{4})$.
- If $V = Y$ then, we choose $U = \{b, c\} \in \mathcal{M}_{\frac{1}{4}}$ such that $f(U) = f(\{b, c\}) = \{y, z\} \subseteq Y = mC(Y, \frac{1}{4})$.

From definition, f is $\frac{1}{4}$-$\mathcal{M}$ weak continuous.

Theorem 3.1.27 Let $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ be a mapping on two r-OSMS's. Then the following statements are equivalent:

1 $f^{-1}(V) \subseteq mI(f^{-1}(mC(V, r)), r)$ for each r-OSM open set V in Y.

2 $mC(f^{-1}(mI(B, r), r) \subseteq f^{-1}(B)$ for each r-OSM closed set B in Y.

Proof. (1) ⇒ (2) Let B be r-OSM closed in Y. By (2) and Theorem 3.1.5 (6),

$$\begin{aligned} X - f^{-1}(B) &= f^{-1}(Y - B) \\ &\subseteq mI(f^{-1}(mC(Y - B, r)), r) \\ &= mI(f^{-1}(Y - mI(B, r)), r) \\ &= mI(X - f^{-1}(mI(B, r)), r) \\ &= X - mC(f^{-1}(mI(B, r)), r). \end{aligned}$$

Thus $X - f^{-1}(B) \subseteq X - mC(f^{-1}(mI(B, r)), r)$.
Hence $mC(f^{-1}(mI(B, r)), r) \subseteq f^{-1}(B)$.
(2) ⇒ (1) Let V be r-OSM open in Y.
Then $Y - V$ is an r-OSM closed set in Y. Therefore

$$\begin{aligned} mC(f^{-1}(mI(Y - V, r)), r) &= mC(f^{-1}(Y - mC(V, r)), r) \\ &= mC(X - f^{-1}(mC(V, r)), r) \\ &= X - mI(f^{-1}(mC(V, r)), r) \\ &\subseteq f^{-1}(Y - V) \\ &= X - f^{-1}(V). \end{aligned}$$

Thus $X - mI(f^{-1}(mC(V, r)), r) \subseteq X - f^{-1}(V)$.
Hence $f^{-1}(V) \subseteq mI(f^{-1}(mC(V, r)), r)$. □

Theorem 3.1.28 Let $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ be a mapping on two r-OSMS's. Then the following statements are hold:

1 If f is r-M weak continuous, then $f^{-1}(V) \subseteq mI(f^{-1}(mC(V, r)), r)$ for each r-OSM open set V in Y.

2 If $mC(f^{-1}(mI(B, r), r) \subseteq f^{-1}(B)$ for each r-OSM closed set B in Y, then $mC(f^{-1}(B), r) \subseteq f^{-1}(mC(B, r))$ for each r-OSM open set B in Y.

3 If $mC(f^{-1}(B),r) \subseteq f^{-1}(mC(B,r))$ for each r-OSM open set B in Y is true and $\mathcal{N}$ has the property $(\mathcal{U})$, then f is r-M weak continuous.

Proof. (1) Let V be r-OSM open in Y, and $x \in f^{-1}(V)$. Then $f(x) \in V$. By f is r-M weak continuous, there exists an r-OSM open set U containing x such that $f(U) \subseteq mC(V,r)$. So $x \in U \subseteq f^{-1}(f(U)) \subseteq f^{-1}(mC(V,r))$ and this implies that $x \in mI(f^{-1}mC(V,r)),r)$. Hence $f^{-1}(V) \subseteq mI(f^{-1}(mC(V,r)),r)$.

(2) Let B be r-OSM open in Y. By Theorem 3.1.5 (2), $mI(B,r) = B$, and by (3), we have $mC(f^{-1}(B),r) = mC(f^{-1}(mI(B,r)),r) \subseteq f^{-1}(B)$. Thus by Theorem 3.1.5 (3), $f^{-1}(B) \subseteq f^{-1}(mC(B,r))$. Hence $mC(f^{-1}(B),r) \subseteq f^{-1}(mC(B,r))$.

(3) Let x be a point set in X and V an r-OSM open set in Y containing $f(x)$ and $\mathcal{N}$ has the property $(\mathcal{U})$. For each $x \in f^{-1}(V)$,

$$\begin{aligned} x \in f^{-1}(V) &\subseteq f^{-1}(mI(mC(V,r),r)) \\ &= X - f^{-1}(mC(Y - mC(V,r),r)) \\ &\subseteq X - mC(f^{-1}(Y - mC(V,r)),r) \\ &= mI(f^{-1}(mC(V,r)),r). \end{aligned}$$

Since $x \in mI(f^{-1}(mC(V,r)),r)$, there exists an r-OSM open set U containing x such that $U \subseteq f^{-1}(mC(V,r))$. Hence$f(U) \subseteq f(f^{-1}(mC(V,r))) \subseteq mC(V,r)$. Therefore f is r-M weak continuous. □

Theorem 3.1.29 Let $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ be a mapping on two r-OSMS's and $A \in 2^Y$. If f is r-M weak continuous, then the following statements hold:

1 $f^{-1}(A) \subseteq mI(f^{-1}(mC(A,r)),r)$ for $A = mI(A,r)$.

2 $mC(f^{-1}(mI(A,r)),r) \subseteq f^{-1}(A)$ for $A = mC(A,r)$.

Proof. (1) Let A be a subset in Y such that $A = mI(A,r)$. Then for each $x \in f^{-1}(A)$, $f(x) \in A = mI(A,r)$. Thus there exists an r-OSM open set V containing $f(x)$ such that $f(x) \in V \subseteq A$. Since f is an r-M weak continuous, there exists an r-OSM open set U containing x such that $f(U) \subseteq mC(V,r)$. Thus $x \in U \subseteq f^{-1}(f(U)) \subseteq f^{-1}(mC(A,r)$. It implies that $x \in mI(f^{-1}(mC(V,r)),r) \subseteq mI(f^{-1}(mC(A,r)),r)$. Hence $f^{-1}(A) \subseteq mI(f^{-1}(mC(A,r)),r)$.

(2) Let $A = mC(A,r)$. Then $Y - A = Y - mC(A,r) = mI(Y - A, r)$.

By (1), we have

$$
\begin{aligned}
X - f^{-1}(A) &= f^{-1}(Y - A) \\
&\subseteq mI(f^{-1}(mC(Y - A, r), r)) \\
&= mI(f^{-1}(Y - mI(A, r)), r) \\
&= mI(X - f^{-1}(mI(A, r), r)) \\
&= X - mC(f^{-1}(mI(A, r), r)).
\end{aligned}
$$

Thus $mC(f^{-1}(mI(A,r),r)) \subseteq f^{-1}(A)$. □

Theorem 3.1.30 Let $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ be an r-M weak continuous mapping on two r-OSMS's. If A is an r-OSM compact set in X and $\mathcal{M}$ has the property $(\mathcal{U})$, then $f(A)$ is an r-OSM almost compact set.

Proof. Let A be an r-OSM compact set in X and $\mathcal{M}$ has property $(\mathcal{U})$ and let $\{B_i \in 2^Y : i \in J\}$ be an r-OSM open cover of $f(A)$ in Y. Then from r-M weak continuity, $f^{-1}(B_i) \subseteq mI(f^{-1}(mC(B_i, r)), r)$ for each $i \in J$ and by Theorem 3.1.15 and $\mathcal{M}$ has the property $(\mathcal{U})$, then $\{mI(f^{-1}(mC(B_i, r)), r) : i \in J\}$ is an r-OSM open cover of A in X. By r-OSM compactness, there exists $J_0 = \{j_1, j_2, ..., j_n\} \subseteq J$ such that $A \subseteq \bigcup_{i \in J_0} mI(f^{-1}(mC(B_i, r)), r) \subseteq f^{-1}(mC(B_i, r))$. Hence $f(A) \subseteq \bigcup_{i \in J_0} mC(B_i, r)$. Therefore $f(A)$ is an r-OSM almost compact set. □

Theorem 3.1.31 Let $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ be an r-M weak continuous and r-M open mapping on two r-OSMS's. If A is an r-OSM almost compact set and $\mathcal{M}$ has the property $(\mathcal{U})$, then $f(A)$ is an r-OSM almost compact set.

Proof. Let A is an r-OSM almost compact set and $\mathcal{M}$ has the property $(\mathcal{U})$ and let $\{B_i \in 2^Y : i \in J\}$ be an r-OSMS open cover of $f(A)$ in Y. Then by the property $(\mathcal{U})$, $\{mI(f^{-1}(mC(B_i, r)), r) : i \in J\}$ is an r-OSM open cover of A in X. So there exists a finite subset $J_0 = \{j_1, j_2, ..., j_n\} \subseteq J$ such that $A \subseteq \bigcup_{i \in J_0} mC(mI(f^{-1}(mC(B_i, r)), r, r)$. From Theorems 3.1.29, 3.1.23 and 3.1.22 , it follows that

$$A \subseteq \bigcup_{i \in J_0} mC(mI(f^{-1}(mC(B_i, r)), r, r)$$

$$\subseteq \bigcup_{i \in J_0} mC(f^{-1}(mI(mC(B_i, r), r)), r)$$
$$\subseteq \bigcup_{i \in J_0} f^{-1}(mC(B_i, r)).$$

Hence $f(A) \subseteq \bigcup_{i \in J_o} mC(B_i, r)$. Therefore $f(A)$ is an r-OSM almost compact set. □

Theorem 3.1.32 Let $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ be an r-M weak continuous and r-M open mapping on two r-OSMS's. If A is an r-OSM nearly r-minimal compact set and $\mathcal{M}$ has property $(\mathcal{U})$, then $f(A)$ is an r-OSM nearly compact set.

Proof. Let A is an r-OSM nearly r-minimal compact set and $\mathcal{M}$ has property $(\mathcal{U})$ and let $\{B_i \in 2^Y : i \in J\}$ be an r-OSMS open cover of $f(A)$ in Y.
Then $\{mI(f^{-1}(mC(B_i, r)), r) : i \in J\}$ is an r-OSM open cover of A in X. By the r-OSM nearly compactness there exists a finite subset $J_0 = \{j_1, j_2, ..., j_n\} \subseteq J$ such that $A \subseteq \bigcup_{i \in J_0} mI(mC(mI(f^{-1}(mC(B_i, r)), r, r, r)$, from Theorems 3.1.29, 3.1.23 and 3.1.22 , it follows

$$A \subseteq \bigcup_{i \in J_0} mI(mC(mI(f^{-1}(mC(B_i, r)), r, r, r)$$
$$\subseteq \bigcup_{i \in J_0} mI(mC(f^{-1}(mI(mC(B_i, r), r)), r, r)$$
$$\subseteq \bigcup_{i \in J_0} mI(f^{-1}(mC(mI(mC(B_i, r), r), r))$$
$$\subseteq \bigcup_{i \in J_o} mI(f^{-1}(mC(B_i, r), r)).$$

Hence $f(A) \subseteq \bigcup_{i \in J_o} mI(mC(B_i, r), r)$. Therefore $f(A)$ is an r-OSM nearly compact set. □

Definition 3.1.33 Let $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ be a mapping two r-OSMS's. Then f is said to be an *ordinary smooth almost* $r - M$ *continuous* (simply, r-M almost continuous) if for $x \in X$ and each r-OSM open set V containing $f(x)$, there is an r-OSM open set U containing x such that $f(U) \subseteq mI(mC(V, r), r)$.

Example 3.1.34 Form Example 3.1.26.
Let $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ and $r = \frac{1}{2}$ define as follows: $f(a) = y = f(b), f(c) = z$.

Then $\mathcal{M}_{\frac{1}{2}} = \{\{c\}, \{b\}, \{b, c\}, \emptyset, X\}, \mathcal{N}_{\frac{1}{2}} = \{\{y\}, \{z\}, \{y, z\}, \emptyset, X\}$.
Consider $a \in X$, $f(a) = y$ is a member of $\frac{1}{2}$-OSM open set $\{y\}, \{y, z\}, Y$.

- If $V = \{y\}$ then, we choose $U = \{b\} \in \mathcal{M}_{\frac{1}{2}}$ such that $f(U) = f(\{b\}) = \{y\} \subseteq \{y\} = mI(mC(\{y\}, \frac{1}{2}), \frac{1}{2})$.
- If $V = \{y, z\}$ then, we choose $U = \{c\} \in \mathcal{M}_{\frac{1}{2}}$. such that $f(U) = f(\{c\}) = \{z\} \subseteq Y = mI(mC(\{y, z\}, \frac{1}{2}), \frac{1}{2})$.
- If $V = Y$ then, we choose $U = \{b, c\} \in \mathcal{M}_{\frac{1}{2}}$ such that $f(U) = f(\{b, c\}) = \{y, z\} \subseteq Y = mI(mC(Y, \frac{1}{2}), \frac{1}{2})$.

Consider $b \in X$, $f(b) = y$ is a member of $\frac{1}{2}$-OSM open set $\{y\}, \{y, z\}, Y$.

- If $V = \{y\}$ then, we choose $U = \{b\} \in \mathcal{M}_{\frac{1}{2}}$ such that $f(U) = f(\{b\}) = \{y\} \subseteq \{y\} = mI(mC(\{y\}, \frac{1}{2}), \frac{1}{2})$.
- If $V = \{y, z\}$ then , we choose $U = \{c\} \in \mathcal{M}_{\frac{1}{2}}$ such that $f(U) = f(\{c\}) = \{z\} \subseteq Y = mI(mC(\{y, z\}, \frac{1}{2}), \frac{1}{2})$.
- If $V = Y$ then, we choose $U = \{b, c\} \in \mathcal{M}_{\frac{1}{2}}$. such that $f(U) = f(\{b, c\}) = \{y, z\} \subseteq Y = mI(mC(Y, \frac{1}{2}), \frac{1}{2})$.

Consider $c \in X$, $f(c) = z$ is a member of $\frac{1}{2}$-OSM open set $\{y, z\}, \{z\}, Y$.

- If $V = \{y, z\}$ then, we choose $U = \{b\} \in \mathcal{M}_{\frac{1}{2}}$ such that $f(U) = f(\{b\}) = \{y\} \subseteq Y = mI(mC(\{y, z\}, \frac{1}{2}), \frac{1}{2})$.
- If $V = \{z\}$ then, we choose $U = \{c\} \in \mathcal{M}_{\frac{1}{2}}$ such that $f(U) = f(\{c\}) = \{z\} \subseteq \{x, z\} = mI(mC(\{z\}, \frac{1}{2}), \frac{1}{2})$.
- If $V = Y$ then, we choose $U = \{b, c\} \in \mathcal{M}_{\frac{1}{2}}$ such that $f(U) = f(\{b, c\}) = \{y, z\} \subseteq Y = mI(mC(Y, \frac{1}{2}), \frac{1}{2})$.

From definition, f is an r-M almost continuous.

Theorem 3.1.35 Let $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ be a mapping on two r-OSMS's. Then the following statements are equivalent:

1 $f^{-1}(B) \subseteq mI(f^{-1}(mI(mC(B,r),r)),r)$ for each r-OSM open set B in Y,

2 $mC(f^{-1}(mC(mI(F,r),r)),r) \subseteq f^{-1}(F)$ for each r-OSM closed set F in Y.

Proof. (1) ⇒ (2) Let F be r-OSM open set in Y. By (2) and Theorem 3.1.5 (6),

$$\begin{aligned} X - f^{-1}(F) &= f^{-1}(Y - F) \\ &\subseteq mI(f^{-1}(mI(mC(Y - F),r),r)),r) \\ &= mI(f^{-1}(mI(Y - mC(mI(F,r),r),r)),r) \\ &= mI(X - f^{-1}(mC(mI(F,r),r)),r) \\ &= X - mC(f^{-1}(mC(mI(F,r),r)),r). \end{aligned}$$

Hence $mC(f^{-1}(mC(mI(F,r),r)),r) \subseteq f^{-1}(F)$.

(2) ⇒ (1) Let B be r-OSM open set in Y. By (3) and Theorem 3.1.5 (6),

$$\begin{aligned} X - f^{-1}(B) = f^{-1}(Y - B) &\supseteq mC(f^{-1}(mC(mI(Y - B,r),r)),r) \\ &= mC(f^{-1}(mC(Y - mC(B,r),r)),r) \\ &= mC(f^{-1}(Y - mI(mC(B,r),r)),r) \\ &= mC(X - f^{-1}(mI(mC(B,r),r)),r) \\ &= X - mI(f^{-1}(mI(mC(B,r),r)),r). \end{aligned}$$

Hence $f^{-1}(B) \subseteq mI(f^{-1}(mI(mC(B,r),r)),r)$. □

Theorem 3.1.36 Let $f : (X,\mathcal{M}) \to (Y,\mathcal{N})$ be a mapping on two r-OSMS's. Then the following statements are hold:

1 If f is r-M almost continuous, then $f^{-1}(B) \subseteq mI(f^{-1}(mI(mC(B,r),r)),r)$ for each r-OSM open set B in Y,

2 If $f^{-1}(B) \subseteq mI(f^{-1}(mI(mC(B,r),r)),r)$ for each r-OSM open set B in Y and $\mathcal{M}$ has the property ($\mathcal{U}$), then f is r-M almost continuous.

Proof. 1. Let B be r-OSM open set in Y, and $x \in f^{-1}(B)$, there exists an r-OSM open set U containing x such that $f(U) \subseteq mI(mC(B,r),r)$. So $x \in U \subseteq f^{-1}(mI(mC(B,r),r)$. This implies that $x \in mI(f^{-1}(mI(mC(B,r),r)),r)$. Hence $f^{-1}(B) \subseteq mI(f^{-1}(mI(mC(B,r),r)),r)$.

2. Let $x \in X$ and V an r-OSM open set containing $f(x)$ and $\mathcal{M}$ has the property $(\mathcal{U})$. Then by (2), $x \in mI(f^{-1}(mI(mC(V,r),r)),r)$, and so there exists an r-OSM open set U containing x such that $U \subseteq f^{-1}(mI(mC(V,r),r))$. we have the following $f(U) \subseteq f(f^{-1}(mI(mC(V,r),r))) \subseteq mI(mC(V,r),r)$.
   Hence f is r-M almost continuous.

□

Theorem 3.1.37 Let $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ be an r-M almost continuous mapping on two r-OSMS's. If A is an r-OSM compact set in X and $\mathcal{M}$ has property $(\mathcal{U})$, then $f(A)$ is an r-OSM nearly compact set.

Proof. Let A is an r-OSM compact set in X and $\mathcal{M}$ has the property $(\mathcal{U})$. Let $\{B_i \in 2^Y : i \in J\}$ be an r-OSM open cover of $f(A)$ in Y. Then from r-M almost continuity, we have $f^{-1}(B_i) \subseteq mI(f^{-1}(mI(mC(B_i,r),r)),r)$ for each $i \in J$. And by Theorem 3.1.5 (1), $\{mI(f^{-1}(mI(mC(B_i,r),r)),r) : i \in J\}$ is an r-OSM open cover of A in X. By the r-OSM compactness, there exists $J_0 = \{j_1, j_2, ..., j_n\} \subseteq J$. such that $A \subseteq \bigcup_{i \in j_0} mI(f^{-1}(mI(mC(B_i,r),r)),r) \subseteq \bigcup_{i \in J_0} f^{-1}(mI(mC(B_i,r),r))$.
Hence $f(A) \subseteq \bigcup_{i \in j_0} mI(mC(B_i,r),r)$. □

Theorem 3.1.38 Let $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ be an r-M almost continuous and r-M open mapping on two r-OSMS's. If A is an r-OSM almost compact set and $\mathcal{M}$ has property $(\mathcal{U})$, then $f(A)$ is an r-OSM almost compact set.

Proof. Let A is an r-OSM almost compact set and $\mathcal{M}$ has property $(\mathcal{U})$. Let $\{B_i \in 2^Y : i \in J\}$ be r-OSM open cover of $f(A)$ in Y. Then $\{mI(f^{-1}(mI(mC(B_i,r)r)),r) : i \in J\}$ is r-OSM open cover of A in X. So there exists a finite subset $J_0 = \{j_1, j_2, ..., j_n\} \subseteq J$ such that $A \subseteq \bigcup_{i \in J_0} mC(mI(f^{-1}(mI(mC(B_i,r)r)),r,r)$. From Theorems 3.1.23 and 3.1.22, it follows

$$
\begin{aligned}
f(A) &\subseteq \bigcup_{i \in J_0} f(mC(mI(f^{-1}(mI(mC(B_i,r)r)),r),r)) \\
&\subseteq \bigcup_{i \in J_0} mC(f^{-1}(mI(mC(B_i,r)r)),r) \\
&\subseteq \bigcup_{i \in J_0} f^{-1}(mC(mI(mC(B_i,r),r)r))
\end{aligned}
$$

$$\subseteq \bigcup_{i\in J_0} f^{-1}(mC(B_i, r).$$

Hence $f(A) \subseteq \bigcup_{i\in J_o} mC(B_i, r)$. □

Theorem 3.1.39 Let $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ be an r-M almost continuous and r-M open mapping on two r-OSMS's. If A is an r-OSM compact set and $\mathcal{M}$ has property $(\mathcal{U})$, then $f(A)$ is an r-OSM compact set.

Proof. Let A is an r-OSM compact set and $\mathcal{M}$ has property $(\mathcal{U})$ and
let $\{B_i \in 2^Y : i \in J\}$ be an r-OSM open cover of $f(A)$ in Y.
Then $\{mI(f^{-1}(mI(mC(B_i, r)r)), r) : i \in J\}$ is an r-OSM open cover of A in X.
So there exists $J_0 = \{j_1, j_2, ..., j_n\} \subseteq J$
such that $A \subseteq \bigcup_{i\in J_0} mI(mC(mI(f^{-1}(mI(mC(B_i, r), r)), r, r, r)$. By the r-OSM nearly compactness. From Theorem 3.1.23 and Theorem 3.1.22, it follows that

$$\begin{aligned}
A &\subseteq \bigcup_{i\in J_0} mI(mC(mI(f^{-1}(mI(mC(B_i, r)r)), r), r), r) \\
&\subseteq \bigcup_{i\in J_0} mI(mC(f^{-1}(mI(mI(mC(B_i, r), r), r)), r), r) \\
&\subseteq \bigcup_{i\in J_0} mI(f^{-1}(mC(mI(mC(B_i, r), r), r)) \\
&\subseteq \bigcup_{i\in J_o} mI(f^{-1}(mC(B_i, r)), r) \\
&\subseteq \bigcup_{i\in J_o} f^{-1}(mI(mC(B_i, r), r)).
\end{aligned}$$

Hence $f(A) \subseteq \bigcup_{i\in J_o} mI(mC(B_i, r), r)$. □

3.2 On Generalized r-mb closed Sets

In this section, we introduce the concept of r-ms closed, r-$mpre$ closed, r-mb closed and r-msp closed in ordinary smooth r-minimal spaces. Characterization some of extremely disconnected spaces and T_{gs} spaces.

Definition 3.2.1 Let $(X, \mathcal{M})$ be an r-OSMS and $A \in 2^X$. Then A is called:

1 ordinary smooth r-minimal semi-closed (briefly r-ms closed)
if $mI(mC(A,r),r) \subseteq A$,

2 ordinary smooth r-minimal pre-closed (briefly r-$mpre$ closed)
if $mC(mI(A,r),r) \subseteq A$,

3 ordinary smooth r-minimal b-closed (briefly r-mb closed)
if $(mC(mI(A,r),r) \cap mI(mC(A,r),r)) \subseteq A$,

4 ordinary smooth r-minimal semi-preclosed (briefly r-msp closed)
if $mI(mC(mI(A,r),r),r) \subseteq A$.

The complement of an r-ms closed (resp. r-$mpre$ closed, r-mb closed, r-msp closed) set is called ordinary smooth r-minimal semi-open (r-ms open) (resp. r-$mpre$ open, r-mb open, r-msp open).

Example 3.2.2 Let $X = \{a, b, c\}$, and $\mathcal{M} : 2^X \to I$.
Let us consider an ordinary smooth r-minimal structure as follws

$$\mathcal{M}(A) = \begin{cases} \frac{3}{4}, & \text{if } A = X, A = \emptyset; \\ \frac{2}{3}, & \text{if } A = \{c\}; \\ \frac{1}{2}, & \text{if } A = \{b\}, \{b,c\}; \\ \frac{1}{4}, & \text{if } A = \{a\}, \{a,b\}, \{a,c\}. \end{cases}$$

Let $r = \frac{1}{2}$, then $\mathcal{M}_{\frac{1}{2}} = \{A \in 2^X : \mathcal{M}(A) \geq \frac{1}{2}\}$. Thus $\mathcal{M}_{\frac{1}{2}} = \{\emptyset, \{b\}, \{b,c\}, \{c\}, X\}$.

(1) Let $A = \{a, b\}$. Then
$mC(\{a,b\}, \frac{1}{2}\}) = \{a,b\}$, $mI(mC(\{a,b\}, \frac{1}{2}), \frac{1}{2}) = \{b\}$,
so $mI(mC(\{a,b\}, \frac{1}{2}), \frac{1}{2}) = \{b\} \subseteq \{a,b\}$. Therefore $\{a,b\}$ is $\frac{1}{2}$-ms closed set.

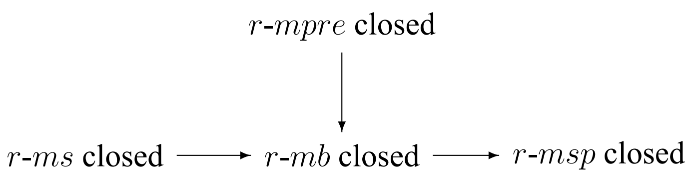
(2) Let $A = \{a, c\}$. Then
$mI(\{a,c\}, \frac{1}{2}\}) = \{c\}$, $mC(mI(\{a,c\}, \frac{1}{2}), \frac{1}{2}) = \{a,c\}$,
so $mC(mI(\{a,c\}, \frac{1}{2}), \frac{1}{2}) = \{a,c\} \subseteq \{a,c\}$. therefore $\{a,c\}$ is $\frac{1}{2}$-$mpre$ closed set.

(3) Let $A = \{b\}$. Then
$mC(mI(\{b\}, \frac{1}{2}), \frac{1}{2}) \cap mI(mC(\{b\}, \frac{1}{2}), \frac{1}{2}) = \{a,b\} \cap \{b\} = \{b\}$,

so $mC(mI(\{b\}, \frac{1}{2}), \frac{1}{2}) \cap mI(mC(\{b\}, \frac{1}{2}), \frac{1}{2}) = \{b\} \subseteq \{b\}$. Therefore $\{b\}$ is $\frac{1}{2}$-mb closed set.

(4) Let $A = \{a, b\}$. then
$mC(mI(\{a, b\}, \frac{1}{2}), \frac{1}{2}) = \{a, b\}$, $mI(mC(mI(\{a, b\}, \frac{1}{2}), \frac{1}{2}), \frac{1}{2}) = \{b\}$,
so $mI(mC(mI(\{a, b\}, \frac{1}{2}), \frac{1}{2}), \frac{1}{2}) = \{b\} \subseteq \{a, b\}$. Therefore $\{a, b\}$ is $\frac{1}{2}$-msp closed set.

It is well-known that:



Lemma 3.2.3 If F is r-$mpre$ closed, then F is r-mb closed.

Proof. Let F be r-$mpre$ closed. Then $mC(mI(F, r), r) \subseteq F$.
Since $mC(mI(F, r), r) \cap mI(mC(F, r), r) \subseteq mC(mI(F, r), r)$, we get that $mC(mI(F, r), r) \cap mI(mC(F, r), r) \subseteq F$. Hence F is r-mb closed. □

Lemma 3.2.4 If F is r-ms closed, then F is r-mb closed.

Proof. Let F be r-ms closed. Then $mI(mC(F, r), r) \subseteq F$.
Since $mC(mI(F, r), r) \cap mI(mC(F, r), r) \subseteq mI(mC(F, r), r)$, we get that $mC(mI(F, r), r) \cap mI(mC(F, r), r) \subseteq F$. Hence F is r-mb closed. □

Lemma 3.2.5 If F is r-mb closed. Then F is r-msp closed.

Proof. Let F be r-mb closed, then $mC(mI(F, r), r) \cap mI(mC(F, r), r) \subseteq F$.
Since $mI(mC(mI(F, r), r), r) \subseteq mC(mI(F, r), r)$ and $mI(mC(mI(F, r), r), r) \subseteq mI(mC(F, r), r)$, we get that $mI(mC(mI(F, r), r), r) \subseteq mC(mI(F, r), r) \cap mI(mC(A, r), r)$. Hence $mI(mC(mI(F, r), r), r) \subseteq F$. Therefore F is r-msp closed. □

Lemma 3.2.6 If F is r-$mpre$ closed, then F is r-msp closed.

Proof. Let F be r-$mpre$ closed. Then $mC(mI(F, r), r) \subseteq F$.
Since $mI(mC(mI(F, r), r), r) \subseteq mC(mI(F, r), r)$, we get that $mI(mC(mI(F, r), r), r) \subseteq F$. Hence F is r-msp closed. □

Lemma 3.2.7 If F is r-ms closed, then F is r-msp closed.

Proof. Let F be r-ms closed. Then $mI(mC(F,r),r) \subseteq F$.
Since $mI(mC(mI(F,r),r),r) \subseteq mI(mC(F,r),r)$, we get that $mI(mC(mI(F,r),r),r)$ $\subseteq F$. Hence F is r-msp closed. □

Definition 3.2.8 Let $(X,\mathcal{M})$ be an r-OSMS and $r \in (0,1]$. We denote the following notators:

1 $smC(A,r) = \cap\{B \in 2^X : B$ is r-ms closed and $A \subseteq B\}$

2 $pmC(A,r) = \cap\{B \in 2^X : B$ is r-$mpre$ closed and $A \subseteq B\}$

3 $bmC(A,r) = \cap\{B \in 2^X : B$ is r-mb closed and $A \subseteq B\}$

4 $spmC(A,r) = \cap\{B \in 2^X : B$ is r-msp closed and $A \subseteq B\}$

Example 3.2.9 (1) Let $\mathcal{M}_{\frac{1}{2}} = \{\emptyset, \{b\}, \{b,c\}, \{c\}, X\}$.

Then $\{a\}, \{b\}, \{a,b\}, \{a,c\}, \{c\}$ are $\frac{1}{2}$-ms closed.

Let $A = \{b\}$, consider,

$$\begin{aligned} mC(\{b\}, \tfrac{1}{2}) &= \cap\{B \in 2^X : B \text{ is } \tfrac{1}{2}\text{-}ms \text{ closed and } \{b\} \subseteq B\} \\ &= \cap\{\{a,b\}, \{b\}\} \\ &= \{b\}. \end{aligned}$$

(2) Let $\mathcal{M}_{\frac{1}{2}} = \{\emptyset, \{b\}, \{b,c\}, \{c\}, X\}$.

Then $\{a\}, \{a,b\}, \{a,c\}$ are $\frac{1}{2}$-$mpre$ closed.

Let $A = \{a,b\}$, consider,

$$\begin{aligned} mC(\{a,b\}, \tfrac{1}{2}) &= \cap\{B \in 2^X : B \text{ is } \tfrac{1}{2}\text{-}mpre \text{ closed and } \{a,b\} \subseteq B\} \\ &= \cap\{\{a,b\}\} \\ &= \{a,b\}. \end{aligned}$$

(3) Let $\mathcal{M}_{\frac{1}{2}} = \{\emptyset, \{b\}, \{b,c\}, \{c\}, X\}$.

Then $\{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}$ are $\frac{1}{2}$-mb closed.

Let $A = \{a\}$, consider,

$$\begin{aligned} mC(\{a\}, \tfrac{1}{2}) &= \cap\{B \in 2^X : B \text{ is } \tfrac{1}{2}\text{-}mb \text{ closed and } \{a\} \subseteq B\} \\ &= \cap\{\{a\}, \{a,b\}, \{a,c\}\} \\ &= \{a\}. \end{aligned}$$

(4) Let $\mathcal{M}_{\frac{1}{2}} = \{\emptyset, \{b\}, \{b,c\}, \{c\}, X\}$.

Then $\{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}$ are $\frac{1}{2}$-msp closed.

Let $A = \{c\}$, consider,

$$\begin{aligned} mC(\{c\}, \tfrac{1}{2}) &= \cap\{B \in 2^X : B \text{ is } \tfrac{1}{2}\text{-}msp \text{ closed and } \{c\} \subseteq B\} \\ &= \cap\{\{a,c\}, \{c\}\} \\ &= \{c\}. \end{aligned}$$

The collection of all r-ms open (resp. r-$mpre$ open, r-mb open, r-msp open) sets of X is denote by r-$mSO(X)$ (resp. r-$mPO(X)$, r-$mBO(X)$, r-$mSPO(X)$) and the collection of all r-ms closed (resp. r-$mpre$ closed, r-mb closed, r-msp closed) sets is denoted by r-$mSC(X)$ (resp. r-$mPC(X)$, r-$mBC(X)$, r-$mSPC(X)$).

Definition 3.2.10 Let $(X, \mathcal{M})$ be an r-OSMS and $A \in 2^X$. Then A is called:

1 r-mgb closed if $bmC(A,r) \subseteq U$ whenever $A \subseteq U$ and $U \in \mathcal{M}_r$.

2 r-msg closed if $smC(A,r) \subseteq U$ whenever $A \subseteq U$ and $U \in r$-$mSO(X)$.

3 r-mgs closed if $smC(A,r) \subseteq U$ whenever $A \subseteq U$ and $U \in \mathcal{M}_r$.

4 r-mgp closed if $pmC(A,r) \subseteq U$ whenever $A \subseteq U$ and $U \in \mathcal{M}_r$.

5 r-$mgsp$ closed if $spmC(A,r) \subseteq U$ whenever $A \subseteq U$ and $U \in \mathcal{M}_r$.

Example 3.2.11 From Example 3.2.2.

(1) Let $r = \frac{1}{2}$ and let $U \in \mathcal{M}_{\frac{1}{2}} = \{\emptyset, \{b\}, \{c\}, \{b,c\}, X\}$.

Then $\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, X$ are $\frac{1}{2}$-mb closed.

Let $A = \{b\}$. Then $A = \{b\} \subseteq \{b,c\} = U$.

Consider,

$$\begin{aligned} bmC(\{b\}, \frac{1}{2}) &= \cap\{F : F \text{ is } \frac{1}{2} - mb \text{ closed and } \{b\} \subseteq F\} \\ &= \cap\{\{b\}, \{a,b\}, X\} \\ &= \{b\} \\ &\subseteq U. \end{aligned}$$

Therefore A is $\frac{1}{2}$-mgb closed.

(2) Let $r = \frac{1}{2}$ and let $U \in \frac{1}{2}\text{-}mSO(X) = \{\emptyset, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, X\}$.

Then $\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, X$ are $\frac{1}{2}$-ms closed.

Let $A = \{c\}$. Then $A = \{c\} \subseteq \{b,c\} = U$.

Consider,

$$\begin{aligned} smC(\{c\}, \frac{1}{2}) &= \cap\{F : F \text{ is } \frac{1}{2} - ms \text{ closed and } \{c\} \subseteq F\} \\ &= \cap\{\{c\}, \{a,c\}, X\} \\ &= \{c\} \\ &\subseteq U. \end{aligned}$$

Therefore A is $\frac{1}{2}$-msg closed.

(3) Let $r = \frac{1}{2}$ and let $U \in \mathcal{M}_{\frac{1}{2}} = \{\emptyset, \{b\}, \{c\}, \{b,c\}, X\}$.

Then $\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, X$ are $\frac{1}{2}$-ms closed.

Let $A = \{b\}$. Then $A = \{b\} \subseteq \{b,c\} = U$.

Consider,

$$\begin{aligned} smC(\{b\}, \frac{1}{2}) &= \cap\{F : F \text{ is } \frac{1}{2} - ms \text{ closed and } \{b\} \subseteq F\} \\ &= \cap\{\{b\}, \{a,b\}, X\} \\ &= \{b\} \\ &\subseteq U. \end{aligned}$$

Therefore A is $\frac{1}{2}$-mgs closed.

(4) Let $r = \frac{1}{2}$ and let $U \in \mathcal{M}_{\frac{1}{2}} = \{\emptyset, \{b\}, \{c\}, \{b,c\}, X\}$.

Then $\emptyset, \{a\}, \{a,b\}, \{a,c\}, X$ are $\frac{1}{2}$-$mpre$ closed.

Let $A = \{c\}$. Then $A = \{c\} \subseteq X = U$.

Consider,

$$\begin{aligned} bmC(\{c\}, \frac{1}{2}) &= \cap\{F : F \text{ is } \frac{1}{2} - mpre \text{ closed and } \{c\} \subseteq F\} \\ &= \cap\{\{a,c\}, X\} \\ &= \{a,c\} \\ &\subseteq U. \end{aligned}$$

Therefore A is $\frac{1}{2}$-mgp closed.

(5) Let $r = \frac{1}{2}$ and let $U \in \mathcal{M}_{\frac{1}{2}} = \{\emptyset, \{b\}, \{c\}, \{b,c\}, X\}$.

Then $\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, X$ are $\frac{1}{2}$-msp closed.

Let $A = \{c\}$. Then $A = \{c\} \subseteq \{b,c\} = U$.

Consider,

$$\begin{aligned} bmC(\{c\}, \frac{1}{2}) &= \cap\{F : F \text{ is } \frac{1}{2} - mb \text{ closed and } \{c\} \subseteq F\} \\ &= \cap\{\{c\}, \{a,c\}, X\} \\ &= \{c\} \\ &\subseteq U. \end{aligned}$$

Therefore A is $\frac{1}{2}$-$mgsp$ closed.

Definition 3.2.12 Let $(X, \mathcal{M})$ be an r-OSMS and $A \in 2^X$. Then A is called **nowhere dense** if and only if $mI(mC(A,r),r) = \emptyset$.

Example 3.2.13 From Example 3.2.2.

Let $A = \{a\}$, consider, $mC(\{a\}, \frac{1}{2}) = \{a\}$, $mI(mC(\{a\}, \frac{1}{2}), \frac{1}{2}) = \emptyset$, therefore $\{a\}$ is nowhere dense.

Definition 3.2.14 Let $(X, \mathcal{M})$ be an r-OSMS and $D \in 2^X$. Then D is called **dense** if and only if $mC(D,r) = X$.

Example 3.2.15 From Example 3.2.2.

Let $D = \{b,c\}$, consider, $mC(\{b,c\}, \frac{1}{2}) = X$, therefore $\{b,c\}$ is dense.

Definition 3.2.16 Let $(X, \mathcal{M})$ be an r-OSMS and $E \in 2^X$. Then E is called **codense** if and only if $mI(E,r) = \emptyset$.

Example 3.2.17 From Example 3.2.2.

Let $E = \{a\}$, consider, $mI(\{a\}, \frac{1}{2}) = \emptyset$, therefore $\{a\}$ is codense.

Definition 3.2.18 An r-OSMS $(X, \mathcal{M})$ is said to be:

1 T_{gs} if every r-mgs closed subset of X is r-msg closed.

2 Extremely disconnected if the r-OSM closure of each r-OSM open subsets of X is r-OSM open.

Example 3.2.19 (1) From Example 3.2.2. Let $A \in 2^X$ and $r = \frac{1}{2}$,

then $\mathcal{M}_{\frac{1}{2}} = \{\emptyset, \{b\}, \{c\}, \{b,c\}, X\}$.

From definition 3.2.1 (1), then $\{a\}, \{b\}, \{a,b\}, \{a,c\}$ and $\{c\}$ are $\frac{1}{2}$-ms closed.

So $\frac{1}{2}$-$mSO(X) = \{\{b,c\}, \{a,c\}, \{c\}, \{b\}, \{a,b\}\}$.

Thus $\{c\}$ and $\{b\}$ is $\frac{1}{2}$-mgs closed. And $\{c\}, \{b\}, \{a,c\}$ and $\{a,b\}$ are $\frac{1}{2}$-msg closed.

Hence every $\frac{1}{2}$-mgs closed subset X is $\frac{1}{2}$-msg closed. Therefore X is T_{gs}.

(2) Let $X = \{a,b,c,d\}$, and $\mathcal{M} : 2^X \to I$.

Let us consider an ordinary smooth $\frac{1}{2}$-minimal structure

$$\mathcal{M}(A) = \begin{cases} \frac{4}{5}, & \text{if } A = X, A = \emptyset; \\ \frac{2}{3}, & \text{if } A = \{a,b,c\}, \{a,b,d\}, \{b,c,d\}, \{a,c,d\}; \\ \frac{1}{2}, & \text{if } A = \{a,b\}, \{a,c\}, \{a,d\}, \{b,c\}, \{b,d\}, \{c,d\}; \\ \frac{1}{4}, & \text{if } A = \{a\}, \{b\}, \{c\}, \{d\}. \end{cases}$$

Let $r = \frac{1}{2}$, then $\mathcal{M}_{\frac{1}{2}} = \{A \in 2^X : \mathcal{M}(A) \geq \frac{1}{2}\}$.

Thus $\mathcal{M}_{\frac{1}{2}} = \{\emptyset, \{a,b\}, \{a,c\}, \{a,d\}, \{b,c\}, \{b,d\}, \{c,d\},$

$\{a,b,c\}, \{a,b,d\}, \{b,c,d\}, \{a,c,d\}, X\}$.

Let A is $\frac{1}{2}$-OSM open.

Let $A = \{a,b\}$, $mC(\{a,b\}, \frac{1}{2}) = \{a,b\}$, so $mC(\{a,b\}, \frac{1}{2})$ is $\frac{1}{2}$-OSM open.

Let $A = \{a,c\}$, $mC(\{a,c\}, \frac{1}{2}) = \{a,c\}$, so $mC(\{a,c\}, \frac{1}{2})$ is $\frac{1}{2}$-OSM open.

Let $A = \{a,d\}$, $mC(\{a,d\}, \frac{1}{2}) = \{a,d\}$, so $mC(\{a,d\}, \frac{1}{2})$ is $\frac{1}{2}$-OSM open.

Let $A = \{b,c\}$, $mC(\{b,c\}, \frac{1}{2}) = \{b,c\}$, so $mC(\{b,c\}, \frac{1}{2})$ is $\frac{1}{2}$-OSM open.

Let $A = \{c,d\}$, $mC(\{c,d\}, \frac{1}{2}) = \{c,d\}$, so $mC(\{c,d\}, \frac{1}{2})$ is $\frac{1}{2}$-OSM open.

Let $A = \{a,b,c\}$, $mC(\{a,b,c\}, \frac{1}{2}) = X$, so $mC(\{a,b,c\}, \frac{1}{2})$ is $\frac{1}{2}$-OSM open.

Let $A = \{a,b,d\}$, $mC(\{a,b,d\}, \frac{1}{2}) = X$, so $mC(\{a,b,d\}, \frac{1}{2})$ is $\frac{1}{2}$-OSM open.

Let $A = \{b,c,d\}$, $mC(\{b,c,d\}, \frac{1}{2}) = X$, so $mC(\{b,c,d\}, \frac{1}{2})$ is $\frac{1}{2}$-OSM open.

Let $A = \{a,c,d\}$, $mC(\{a,c,d\}, \frac{1}{2}) = X$, so $mC(\{a,c,d\}, \frac{1}{2})$ is $\frac{1}{2}$-OSM open.

Therefore X is Extremely disconnected.

Definition 3.2.20 Let $(X, \mathcal{M})$ be an r-OSMS and $A \in 2^X$. Then A is called :

1 ordinary smooth r-minimal semi-open (briefly r-ms open)
if $A \subseteq mC(mI(A,r), r)$,

2 ordinary smooth r-minimal regular open (briefly r-mrg open)

if $A = mI(mC(A,r),r)$,

3 ordinary smooth r-minimal pre-open (briefly r-$mpre$ open) if $A \subseteq mI(mC(A,r),r)$.

The complement of an r-ms open (resp. r-$mpre$ open, r-mrg open) set is called ordinary smooth r-minimal semi-open (r-ms closed) (resp. r-$mpre$ closed, r-mrg closed).

Example 3.2.21 From Example 3.2.2.

(1) Since Example 3.2.2(1), then $\{a,b\}$ is $\frac{1}{2}$-ms closed, thus $\{c\}$ is r-ms open.

(2) Let $A = \{b\}$. Consider, $mC(\{b\},\frac{1}{2}\}) = \{a,b\}$, $mI(mC(\{b\},\frac{1}{2}),\frac{1}{2}) = \{b\}$. So $\{b\} \subseteq mI(mC(\{b\},\frac{1}{2}),\frac{1}{2})$. Therefore $\{b\}$ is $\frac{1}{2}$-mrg open.

(3) Since Example 3.2.2(2), $\{a,c\}$ is r-$mpre$ closed. Thus $\{b\}$ is $\frac{1}{2}$-$mpre$ open.

Lemma 3.2.22 If A_α is r-ms closed for all $\alpha \in \Lambda$, then $\bigcap\limits_{\alpha\in\Lambda} A_\alpha$ is r-ms closed.

Proof. For $\alpha \in \Lambda$, let A_α be r-ms closed. Thus $mI(mC(A_\alpha,r),r) \subseteq A_\alpha$ for all $\alpha \in \Lambda$. Therefore, $mI(mC(\bigcap\limits_{\alpha\in\Lambda} A_\alpha,r),r) \subseteq mI(mC(A_\alpha,r),r) \subseteq A_\alpha$ for all $\alpha \in \Lambda$. Thus $mI(mC(\bigcap\limits_{\alpha\in\Lambda} A_\alpha,r),r) \subseteq \bigcap\limits_{\alpha\in\Lambda} A_\alpha$. Hence $\bigcap\limits_{\alpha\in\Lambda} A_\alpha$ is r-msp closed. $\square$

Lemma 3.2.23 If A_α is an r-mb closed set for $\alpha \in \Lambda$, then $\bigcap\limits_{\alpha\in\Lambda} A_\alpha$ is r-mb closed.

Proof. For $\alpha \in \Lambda$, let A_α be an r-mb closed set.
Thus $mC(mI(A_\alpha,r),r) \cap mI(mC(A_\alpha,r),r) \subseteq A_\alpha$ for all $\alpha \in \Lambda$.
Therefore, for $\alpha \in \Lambda$, $mC(mI(\bigcap\limits_{\alpha\in\Lambda} A_\alpha,r),r) \cap mI(mC(\bigcap\limits_{\alpha\in\Lambda} A_\alpha,r),r)$ $\subseteq mC(mI(A_\alpha,r),r) \cap mI(mC(A_\alpha,r),r)$. And so $mC(mI(\bigcap\limits_{\alpha\in\Lambda} A_\alpha,r),r)$ $\cap\, mI(mC(\bigcap\limits_{\alpha\in\Lambda} A_\alpha,r),r) \subseteq A_\alpha$ for all $\alpha \in \Lambda$. Thus $mC(mI(\bigcap\limits_{\alpha\in\Lambda} A_\alpha,r),r)$ $\cap\, mI(mC(\bigcap\limits_{\alpha\in\Lambda} A_\alpha,r),r) \subseteq \bigcap\limits_{\alpha\in\Lambda} A_\alpha$. Hence $\bigcap\limits_{\alpha\in\Lambda} A_\alpha$ is r-mb closed. $\square$

Lemma 3.2.24 If A_α is r-msp closed for all $\alpha \in \Lambda$, then $\bigcap\limits_{\alpha\in\Lambda} A_\alpha$ is r-msp closed.

Proof. For $\alpha \in \Lambda$, let A_α be r-msp closed. Thus $mI(mC(mI(A_\alpha, r), r), r) \subseteq A_\alpha$ for all $\alpha \in \Lambda$. Therefore, $mI(mC(mI(\bigcap_{\alpha\in\Lambda} A_\alpha, r), r), r) \subseteq mI(mC(mI(A_\alpha, r), r), r) \subseteq A_\alpha$ for all $\alpha \in \Lambda$. This implies that $mI(mC(mI(\bigcap_{\alpha\in\Lambda} A_\alpha, r), r), r) \subseteq \bigcap_{\alpha\in\Lambda} A_\alpha$.
Hence $\bigcap_{\alpha\in\Lambda} A_\alpha$ is r-msp closed. □

Lemma 3.2.25 If A_α is r-$mpre$ closed for all $\alpha \in \Lambda$, then $\bigcap_{\alpha\in\Lambda} A_\alpha$ is r-$mpre$ closed.

Proof. Let A_α is r-$mpre$ closed for all $\alpha \in \Lambda$. For any $\alpha \in \Lambda$, we have $mC(mI(A_\alpha, r), r) \subseteq A_\alpha$. Therefore $mC(mI(\bigcap_{\alpha\in\Lambda} A_\alpha, r), r) \subseteq mC(mI(A_\alpha, r), r)$ for all $\alpha \in \Lambda$.
And so $mC(mI(\bigcap_{\alpha\in\Lambda} A_\alpha, r), r) \subseteq A_\alpha$ for all $\alpha \in \Lambda$, then $mC(mI(\bigcap_{\alpha\in\Lambda} A_\alpha, r), r) \subseteq \bigcap_{\alpha\in\Lambda} A_\alpha$. Hence $\bigcap_{\alpha\in\Lambda} A_\alpha$ is r-$mpre$ closed. □

Lemma 3.2.26 If $A \subseteq B$, then $pmC(A, r) \subseteq pmC(B, r)$.

Proof. Let $A \subseteq B$. Suppose $x \notin pmC(B, r)$, there exists F is r-$mpre$ closed which $B \subseteq F$ but $x \notin F$. Since $A \subseteq B$, we have $A \subseteq F$. Therefore $x \notin pmC(A, r)$.
Hence $pmC(A, r) \subseteq pmC(B, r)$. □

Lemma 3.2.27 Let $(X, \mathcal{M})$ be an r-OSMS and $A \in 2^X$. Then the following statements are hold:

1 $pmC(A, r) = A \cup mC(mI(A, r), r)$,

2 $smC(A, r) = A \cup mI(mC(A, r), r)$.

Proof. (1) Since by Lemma 3.2.25, $pmC(A, r)$ is r-$mpre$ closed.
Then $mC(mI(pmC(A, r), r), r) \subseteq pmC(A, r)$.
Therefore $mC(mI(A, r), r) \subseteq pmC(A, r)$,
and so $A \cup mC(mI(A, r), r) \subseteq pmC(A, r)$.
We show that $pmC(A, r) \subseteq A \cup mC(mI(A, r), r)$. Consider,
$mC(mI(A \cup mC(mI(A, r), r), r), r)$
$= mC(mI(A, r) \cup mI(mC(mI(A, r), r), r), r)$
$\subseteq mC(mI(A, r), r) \cup mC(mI(mC(mI(A, r), r), r), r)$
$= mC(mI(A, r), r)$.

Thus $mC(mI(A \cup mC(mI(A,r),r),r),r) \subseteq mC(mI(A,r),r) \subseteq$
$A \cup mC(mI(A,r),r)$. Hence $A \cup mC(mI(A,r),r)$ is r-$mpre$ closed, and so
$pmC(A,r) \subseteq pmC(A \cup mC(mI(A,r),r),r) = A \cup mC(mI(A,r),r)$.
Therefore $pmC(A,r) = A \cup mC(mI(A,r),r)$.

(2) Since by Lemma 3.2.22, $smC(A,r)$ is r-ms closed.
Then $mI(mC(smC(A,r),r),r) \subseteq smC(A,r)$.
Therefore $mI(mC(A,r),r) \subseteq smC(A,r)$,
and so $A \cup mI(mC(A,r),r) \subseteq smC(A,r)$.
We show that $smC(A,r) \subseteq A \cup mI(mC(A,r),r)$. Consider,
$mI(mC(A \cup mI(mC(A,r),r),r),r)$
$= mI(mC(A,r) \cup mC(mI(mC(A,r),r),r),r)$
$\subseteq mI(mC(A,r),r) \cup mI(mC(mI(mC(A,r),r),r),r)$
$= mI(mC(A,r),r) \cup mI(mC(A,r),r)$
$= mI(mC(A,r),r)$.
Thus $mI(mC(A \cup mI(mC(A,r),r),r),r) \subseteq mI(mC(A,r),r) \subseteq$
$A \cup mI(mC(A,r),r)$. Hence $A \cup mI(mC(A,r),r)$ is r-ms closed, and so
$smC(A,r) \subseteq smC(A \cup mI(mC(A,r),r),r) = A \cup mI(mC(A,r),r)$.
Therefore $smC(A,r) = A \cup mI(mC(A,r),r)$. □

Lemma 3.2.28 If G is an r-OSM open set, then G is r-ms open.

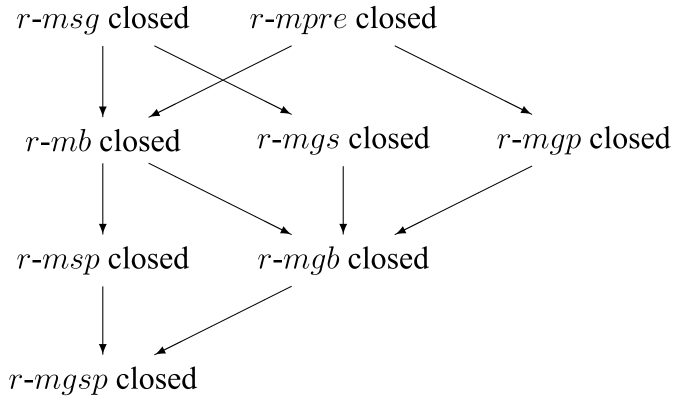
Proof. Let G be r-OSM open, then $G = mI(G,r)$
Since $G \subseteq mC(G,r)$, we have $G \subseteq mC(mI(G,r),r)$.
Therefore G is r-ms open. □

Lemma 3.2.29 If G is r-mrg open and $mI(G,r)$ is r-OSM open, then G is r-OSM open.

Proof. Let G be r-mrg open, then $G = mI(mC(G,r),r)$.
Thus $mI(G,r) = mI(mI(mC(G,r),r),r) = mI(mC(G,r),r) = G$.
This implies that G is r-OSM open. □

3.3 r-mgb Closed Sets and Their Relationships

The relationships between various types of generalized closed sets have been summarized in the following diagram. None of the implications shown in this diagram can be reversed in general.



Lemma 3.3.1 Every r-msg closed set is r-mgs closed.

Proof. Let A be r-msg closed and let $U \in \mathcal{M}_r$ be such that $A \subseteq U$. By Lemma 3.2.28, we have U is r-ms open. Since A is r-msg closed, we get that $smC(A, r) \subseteq U$. Therefore A is r-mgs closed. Hence every r-msg closed set is r-mgs closed. □

Lemma 3.3.2 Every r-$mpre$ closed set is r-mgp closed.

Proof. Let A be r-$mpre$ closed and let $U \in \mathcal{M}_r$ be such that $A \subseteq U$.
Then $pmC(A, r) = \cap\{F \in 2^X : F$ is r-$mpre$ closed and $A \subseteq F\} = A$.
Thus $pmC(A, r) \subseteq U$. Therefore A is r-mgp closed.
Hence every r-$mpre$ closed set is r-mgp closed. □

Lemma 3.3.3 Every r-mb closed set is r-mgb closed.

Proof. Let A be r-mb closed and let $U \in \mathcal{M}_r$ be such that $A \subseteq U$.
Then $bmC(A, r) = \cap\{F \in 2^X : F$ is r-mb closed and $A \subseteq F\} = A$.
Thus $bmC(A, r) \subseteq U$. Therefore A is r-mgb closed.
Hence every r-mb closed set is r-mgb closed. □

Lemma 3.3.4 Every r-msp closed set is r-$mgsp$ closed.

Proof. Let A be r-msp closed and let $U \in \mathcal{M}_r$ be such that $A \subseteq U$.
Then $spmC(A, r) = \cap\{F \in 2^X : F$ is r-msp closed and $A \subseteq F\} = A$.
Thus $spmC(A, r) \subseteq U$. Therefore A is r-$mgsp$ closed.
Hence every r-msp closed set is r-$mgsp$ closed. □

Lemma 3.3.5 Every r-mgs closed set is r-mgb closed.

Proof. Let A be r-mgs closed and let $U \in \mathcal{M}_r$ be such that $A \subseteq U$.
Since A is r-mgs closed, we have

$$\begin{aligned} smC(A,r) &= \cap\{F \in 2^X : F \text{ is } r-ms \text{ closed and } A \subseteq F\}, \\ &\supseteq \cap\{F \in 2^X : F \text{ is } r-mb \text{ closed and } A \subseteq F\}, \\ &= bmC(A,r), \\ &\subseteq U. \end{aligned}$$

Thus $bmC(A,r) \subseteq U$. Therefore A is r-mgb closed □

Lemma 3.3.6 Every r-mgb closed set is r-$mgsp$ closed.

Proof. Let A be r-mgb closed and let $U \in \mathcal{M}_r$ be such that $A \subseteq U$.
Since A is r-mgb closed, we have

$$\begin{aligned} bmC(A,r) &= \cap\{F \in 2^X : F \text{ is } r-mb \text{ closed and } A \subseteq F\}, \\ &\supseteq \cap\{F \in 2^X : F \text{ is } r-msp \text{ closed and } A \subseteq F\}, \\ &= spmC(A,r), \\ &\subseteq U. \end{aligned}$$

Thus $spmC(A,r) \subseteq U$. Therefore A is r-$mgsp$ closed □

Lemma 3.3.7 Every r-mgp closed set is r-mgb closed.

Proof. Let A be r-mgp closed and let $U \in \mathcal{M}_r$ be such that $A \subseteq U$.
Since A is r-mgp closed, we have

$$\begin{aligned} pmC(A,r) &= \cap\{F \in 2^X : F \text{ is } r-mpre \text{ closed and } A \subseteq F\}, \\ &\supseteq \cap\{F \in 2^X : F \text{ is } r-mb \text{ closed and } A \subseteq F\}, \\ &= bmC(A,r), \\ &\subseteq U. \end{aligned}$$

Thus $bmC(A,r) \subseteq U$. Therefore A is r-mgb closed □

Lemma 3.3.8 Every r-ms closed set is r-msg closed.

Proof. Let A be r-ms closed and let $U \in r$-$mSO(X)$ be such that $A \subseteq U$.
Then $smC(A,r) = \cap\{F \in 2^X : F \text{ is } r\text{-}ms \text{ closed and } A \subseteq F\} = A$.

Thus $smC(A, r) \subseteq U$. Therefore A is r-msg closed.

Hence every r-ms closed set is r-msg closed. □

Lemma 3.3.9 Every r-mgp closed set is r-$mgsp$ closed.

Proof. Let A be r-mgp closed and let $U \in \mathcal{M}_r$ be such that $A \subseteq U$.

Then $pmC(A, r) \subseteq U$. Thus by Lemma 3.2.25, $pmC(A, r)$ is r-$mpre$ closed.

By Lemma 3.2.6, $pmC(A, r)$ is r-msp closed. And by Lemma 3.3.4, $pmC(A, r)$ is r-$mgsp$ closed. Hence every r-mgp closed set is r-$mgsp$ closed. □

Lemma 3.3.10 Every r-ms closed set is r-mgs closed.

Proof. Let A be r-ms closed and let $U \in \mathcal{M}_r$ be such that $A \subseteq U$.

Then $smC(A, r) = A \subseteq U$. Therefore A is r-mgs closed □

The main aim of our paper is to investigate more characterizations and properties of the classes of space where the converses hold.

Theorem 3.3.11 Let $(X, \mathcal{M})$ be an r-OSMS. Then the following statements are equivalent:

1 Every r-mgb closed set is r-mgp closed.

2 Every r-mb closed set is r-mgp closed.

Proof. (1) ⇒ (2) Let A be r-mb closed, then by Lemma 3.3.3, A is r-mgb closed, and so by (1), A is r-mgp closed. Therefore every r-mb closed is r-mgp closed.

(2) ⇒ (1) Let A be r-mgb closed and let $U \in \mathcal{M}_r$ be such that $A \subseteq U$. Then $bmC(A, r) \subseteq U$. Thus, by Lemma 3.2.23, $bmC(A, r)$ is r-mb closed, and so by (2), $bmC(A, r)$ is r-mgp closed, so $pmC(A, r) \subseteq pmC(bmC(A, r), r) \subseteq U$. Thus $pmC(A, r) \subseteq U$. Hence A is r-mgp closed. Therefore every r-mgb closed is r-mgp closed. □

Lemma 3.3.12 Let $(X, \mathcal{M})$ be r-OSMS and $\mathcal{M}$ has the property $(\mathcal{U})$. Then the following statements are equivalent:

1 X is extremely disconnected,

2 If G is r-mrg open, then G is r-OSM closed.

Proof. (1) $\Rightarrow$ (2) Let G be r-mrg open, by Lemma 3.2.29, $G = mI(mC(G,r),r)$ is r-OSM open. By (1), $mC(mI(mC(G,r),r),r)$ is r-OSM open. Then

$$\begin{aligned} mC(G,r) &= mC(mI(mC(G,r),r),r) \\ &= mI(mC(mI(mC(G,r),r),r),r) \\ &= mI(mC(G,r),r) \\ &= G. \end{aligned}$$

Therefore G is r-OSM closed.

(2) $\Rightarrow$ (1) Let G be r-OSM open and $mI(G,r) = G$,

then $G \subseteq mC(G,r)$, $mI(G,r) \subseteq mI(mC(G,r),r)$ and $G \subseteq mI(mC(G,r),r)$.

Thus $mC(G,r) \subseteq mC(mI(mC(G,r),r),r)$.

So $X - mC(mI(mC(G,r),r),r) \subseteq X - mC(G,r)$. (*)

And from $mC(mI(mC(A,r),r),r) \subseteq mC(mC(mC(G,r),r),r) = mC(G,r)$.

Then $X - mC(G,r) \subseteq X - mC(mI(mC(G,r),r),r)$. (**)

Form (*) and (**), then $X - mC(G,r) = X - mC(mI(mC(G,r),r),r)$. Therefore

$$\begin{aligned} mI(X-G,r) &= mI(X - mI(mC(G,r),r),r) \\ &= mI(mC(X - mC(G,r),r),r) \\ &= mI(mC(mI(X-G,r),r),r). \end{aligned}$$

Thus $mI(X-G,r)$ is r-mrg open. By (2), $mI(X-G,r)$ is r-OSM closed.

This implies $X - mC(G,r) = mI(X-G,r)$ is r-OSM closed.

And so $mC(G,r)$ is r-OSM open. Hence X is extremely disconnected. □

Theorem 3.3.13 Let $(X, \mathcal{M})$ be r-OSMS and $\mathcal{M}$ has the property $(\mathcal{U})$. Then the following statements are equivalent:

1 Every r-$mgsp$ closed set is r-mgp closed.

2 Every r-msp closed set is r-mgp closed.

3 Every r-mgs closed set is r-mgp closed.

4 Every r-msg closed set is r-mgp closed.

5 Every r-msp closed is r-$mpre$ closed.

6 Every r-mgb closed set is r-mgp closed.

7 Every r-mb closed set is r-mgp closed.

8 X is extremely disconnnected.

Proof. (1) ⇒ (2) Let A be r-msp closed,
then by Lemma 3.3.4, A is r-$mgsp$ closed, and so by (1) , A is r-mgp closed.
Therefore every r-msp closed set is r-mgp closed.
(2) ⇒ (1) Let A be r-$mgsp$ closed and let $U \in \mathcal{M}_r$ be such that $A \subseteq U$.
Then $spmC(A, r) \subseteq U$. Thus by Lemma 3.2.24, $spmC(A, r)$ is r-msp closed,
and so by (2), $spmC(A, r)$ is r-mgp closed.
This implies $pmC(A, r) \subseteq pmC(spmC(A, r), r) \subseteq U$.
Therefore A is r-mgp closed.
(2) ⇒ (3) Let A be r-mgs closed and let $U \in \mathcal{M}_r$ be such that $A \subseteq U$.
Then $smC(A, r) \subseteq U$. Thus by Lemma 3.2.22, $smC(A, r)$ is r-ms closed.
By Lemma 3.2.7, $smC(A, r)$ is r-msp closed,
and so by (2), $smC(A, r)$ is r-mgp closed. This implies,
$pmC(A, r) \subseteq pmC(smC(A, r), r) \subseteq U$. Hence A is r-mgp closed.
(3) ⇒ (4) Let A be r-msg closed, then by Lemma 3.3.1, A is r-mgs closed.
By (3), A is r-mgp closed.
(4) ⇒ (8) Let A be r-mrg open. Then $A = mI(mC(A, r), r)$.
Therefore A is r-ms closed. By Lemma 3.3.8, A is r-msg closed.
And so by (4), A is r-mgp closed. Since $A = mI(mC(A, r), r)$ and $mI(mC(A, r), r)$ is r-OSM open set. Then $pmC(A, r) \subseteq mI(mC(A, r), r) = A$. This implies, $A \subseteq A \cup mC(mI(A, r), r) = pmC(A, r) \subseteq A$, and so $mC(mI(A, r), r) \subseteq A$. Consider,

$$
\begin{aligned}
A &\subseteq mC(A, r) \\
&= mC(mI(mC(A, r), r), r) \\
&= mC(mI(mI(mC(A, r), r), r), r) \\
&= mC(mI(A, r), r) \\
&\subseteq A.
\end{aligned}
$$

Thus $A = mC(A, r)$. Hence A is r-OSM closed. Therefore X is extremely disconnnected.

(8) ⇒ (2) Let A be r-msp closed, then $mI(mC(mI(A,r),r),r) \subseteq A$. Since $mI(A,r)$ is r-OSM open and X is extremely disconnected, we have $mC(mI(A,r),r)$ is r-OSM open. Since $mI(mC(mI(A,r),r),r) = mC(mI(A,r),r)$ and $mI(mC(mI(A,r),r),r) \subseteq A$, we get that $mC(mI(A,r),r) \subseteq A$. Hence A is r-$mpre$ closed. Therefore by Lemma 3.3.2, A is r-mgp closed.

(5) ⇒ (8) Let A be r-mrg open. Then $A = mI(mC(A,r),r)$. Since $A \subseteq mC(A,r)$, we have $mI(mC(mI(A,r),r),r) \subseteq mI(mC(mI(mC(A,r),r),r),r) = mI(mC(A,r),r)$ $= A$. So A is r-msp closed. By (5), A is r-$mpre$ closed. Thus $mC(mI(A,r),r) \subseteq A$. Therefore

$$\begin{aligned} mC(mI(A,r),r) &= mC(mI(mI(mC(A,r),r),r),r) \\ &= mC(mI(mC(A,r),r),r) \\ &= mC(A,r). \end{aligned}$$

Since $mC(mI(A,r),r) \subseteq A$, we have $mC(A,r) \subseteq A$ and $mC(A,r) = A$.
Hence A is r-OSM closed. Therefore X is extremely disconnected.

(8) ⇒ (5) Let A be r-msp closed, then $mI(mC(mI(A,r),r),r) \subseteq A$.
Since $mI(A,r)$ is r-OSM open and X is extremely disconnected, then $mC(mI(A,r),r)$ is r-OSM open, we get that $mI(mC(mI(A,r),r),r) = mC(mI(A,r),r)$.
So $mC(mI(A,r),r) \subseteq A$. Hence A is r-$mpre$ closed.

(6) ⇔ (7) Proved in Theorem 3.3.11.

(6) ⇒ (8) Let A be r-mrg open. Then $A = mI(mC(A,r),r)$.
Since $mI(mC(A,r),r) \cap mC(mI(A,r),r) \subseteq mI(mC(A,r),r) = A$,
we have A is r-mb closed, By Lemma 3.3.3, A is r-mgb closed, and by (6), A is r-mgp closed. Since $A = mI(mC(A,r),r), pmC(A,r) \subseteq mI(mC(A,r),r) = A$. This implies $A \subseteq A \cup mC(mI(A,r),r) = pmC(A,r) \subseteq A$ and so $mC(mI(A,r),r) \subseteq A$. Consider,

$$\begin{aligned} A &\subseteq mC(A,r) \\ &= mC(mI(mC(A,r),r),r) \\ &= mC(mI(mI(mC(A,r),r),r),r) \\ &= mC(mI(A,r),r) \\ &\subseteq A \end{aligned}$$

Thus $A = mC(A, r)$. Hence A is r-OSM closed.

Therefore X is extremely disconnected.

(8) ⇒ (7) Let A be r-mb closed. Since $mI(mC(A, r), r)$ is r-OSM open, and X is extremely disconnected, we have $mC(mI(mC(A, r), r), r)$ is r-OSM open. Thus $mI(mC(mI(mC(A, r), r), r), r) = mC(mI(mC(A, r), r), r)$. Consider,

$$\begin{aligned} mI(A, r) &\subseteq mC(A, r) \\ mI(mI(A, r), r) &\subseteq mI(mC(A, r), r) \\ mI(A, r) &\subseteq mI(mC(A, r), r) \\ mC(mI(A, r), r) &\subseteq mC(mI(mC(A, r), r), r) \\ &= mI(mC(mI(mI(mC(A, r), r), r), r), r) \\ &\subseteq mI(mC(mC(mC(A, r), r), r), r) \\ &= mI(mC(A, r), r). \end{aligned}$$

Since A is r-mb closed, we have $mC(mI(A, r), r) = mC(mI(A, r), r) \cap mI(mC(A, r), r)$ $\subseteq A$. Thus A is r-$mpre$ closed, and so by Lemma 3.3.2, A is r-mgp closed. □

Definition 3.3.14 Let $(X, \mathcal{M})$ be r-OSMS and let $X_1, X_2 \subseteq X$ defined by $X_1 = \{x \in X : \{x\}$ is nowhere dense$\}$ and $X_2 = \{x \in X : \{x\}$ is r-$mpre$ open$\}$. It is easy to see that $\{X_1, X_2\}$ is a decomposition of X (i.e. $X = X_1 \cup X_2$).

Example 3.3.15 From Example 3.2.13, then $\{a\}$ is nowhere dense, so $X_1 = \{a\}$. And from Example 3.2.2, then $\{b, c\}$ is r-$mpre$ open, so $X_2 = \{b, c\}$. Hence $X = \{a, b, c\} = X_1 \cup X_2$. Therefore $\{a, b, c\}$ is a decomposition of X.

Theorem 3.3.16 Let $(X, \mathcal{M})$ be r-OSMS and $A \in 2^X$. Then A is r-msg closed if and only if $X_1 \cap smC(A, r) \subseteq A$.

Proof. (⇒) Let $x \in X_1 \cap smC(A, r)$, then $\{x\}$ is r-ms closed. Assume that $x \notin A$, then $A \subseteq X - \{x\}$. Thus $smC(A, r) \subseteq X - \{x\}$, contradicts. Therefore $x \in A$. Hence $X_1 \cap smC(A, r) \subseteq A$.

(⇐) Suppose that $X_1 \cap smC(A, r) \subseteq A$. Let $U \in r$-$mSO(X)$ be such that $A \subseteq U$ and let $x \in smC(A, r)$.

If $x \in X_1$, then $x \in X_1 \cap smC(A, r) \subseteq A$. So $smC(A, r) \subseteq A \subseteq U$.

If $x \in X_2$, then suppose that $x \notin U$. This implies that $X - U$ is r-ms closed and $x \in X - U$. Since $\{x\}$ is r-$mpre$ open, we have

$$\begin{aligned} smC(\{x\}, r) &= \{x\} \cup mI(mC(\{x\}, r), r) \\ &= mI(mC(\{x\}, r), r), r) \\ &\subseteq mI(mC(X - U, r), r) \\ &\subseteq X - U \\ &\subseteq X - A. \end{aligned}$$

So $mI(mC(\{x\}, r), r) \cap A = \emptyset$ and $A \subseteq X - mI(mC(\{x\}, r), r) = mC(mI(X - \{x\}, r), r)$. Consider

$$\begin{aligned} mI(mC(X - mI(mC(\{x\}, r), r), r), r) &= X - mC(mI(mI(mC(\{x\}, r), r), r), r) \\ &= X - mC(mI(mC(\{x\}, r), r), r) \\ &\subseteq X - mC(\{x\}, r) \\ &\subseteq X - mI(mC(\{x\}, r), r). \end{aligned}$$

Thus $X - mI(mC(\{x\}, r), r)$ is r-ms closed.

Since $x \in smC(A, r)$, we have $x \in X - mI(mC(\{x\}, r), r) \subseteq X - \{x\}$, contradition.

Thus $x \in U$ and $smC(A, r) \subseteq U$. Hence A is r-msg closed. □

Lemma 3.3.17 Let $(X, \mathcal{M})$ be an r-OSMS, X is T_{gs} if and only if every singleton is either r-$mpre$ open or r-OSM closed.

Proof. (⇒) Let $x \in X$, then $x \in X_1$ or $x \in X_2$.

If $x \in X_1$. Assume that $\{x\}$ is not r-OSM closed.

Then $X - \{x\}$ is not r-OSM open, and $X - \{x\}$ is r-mgs closed. If follows from $x \in X_1$ that $X - \{x\}$ is dense and $smC(X - \{x\}, r) = X$. Since X is T_{gs}, then $X - \{x\}$ is r-msg closed. By Theorem 3.3.16, then $X_1 = X_1 \cap smC(X - \{x\}, r) \subseteq X - \{x\}$, contradicts. Hence $\{x\}$ is r-OSM closed.

If $x \in X_2$, then X is r-$mpre$ open.

(⇐) Let A be r-mgs closed. We show that $X_1 \cap smC(A, r) \subseteq A$. Let $x \in X_1 \cap smC(A, r)$, then $\{x\}$ is r-OSM closed. Assume that $x \notin A$, then $A \subseteq X - \{x\}$. Thus $smC(A, r) \subseteq X - \{x\}$, contradicts. Hence $x \in A$. Therefore$A$ is a r-msg closed. □

Lemma 3.3.18 Let $(X, \mathcal{M})$ be an r-OSMS, every singleton is either r-$mpre$ open or nowhere dense.

Proof. Let $x \in X$. Assume that $\{x\}$ is not nowhere dense, then $mI(mC(\{x\}, r), r) \neq \emptyset$. Assume that $x \notin mI(mC(\{x\}, r), r)$, $x \notin G_\alpha$ for all $\alpha \in \Lambda$ which $G_\alpha \in \mathcal{M}_r$ and $G_\alpha \subseteq mC(\{x\}, r)$. Therefore $G_\alpha \subseteq mC(\{x\}, r) \subseteq mC(X - G_\alpha, r) = X - G_\alpha$ for all $\alpha \in \Lambda$, contradicts. Thus $x \in mI(mC(\{x\}, r), r)$. So $\{x\} \subseteq mI(mC(\{x\}, r), r)$.
Hence $\{x\}$ is r-$mpre$ open. □

Theorem 3.3.19 Let $(X, \mathcal{M})$ be an r-OSMS, the following statements are equivalent:

1 Every r-mgb closed set is r-mb closed.

2 Every r-mgs closed set is r-mb closed.

3 Every r-mgb closed set is r-msp closed.

4 Every r-mgp closed set is r-$mpre$ closed.

5 Every r-$mgsp$ closed set is r-msp closed.

6 Every r-mgp closed set is r-msp closed.

7 X is T_{gs}.

Proof. (1) ⇒ (2) Let A be r-mgs closed and $U \in \mathcal{M}_r$ be such that $A \subseteq U$, then $smC(A, r) \subseteq U$. By Lemma 3.2.22, $smC(A, r)$ is r-ms closed, and by Lemma 3.2.4, $smC(A, r)$ is r-mb closed, and so by Lemma 3.3.3, $smC(A, r)$ is r-mgb closed. Therefore $bmC(A, r) \subseteq bmC(smC(A, r), r) \subseteq U$. Thus A is r-mgb closed. Hence by (1), A is r-mb closed.
(2) ⇒ (7) Let $x \in X$, then by Lemma 3.3.18, $\{x\}$ is either r-$mpre$ open or nowhere dense.
Case 1 : If $\{x\}$ is r-$mpre$ open, then by Lemma 3.3.17, X is T_{gs}.
Case 2 : If $\{x\}$ is nowhere dense.
We show that $\{x\}$ is r-OSM closed.
Assume that $\{x\}$ is not r-OSM closed, then $X - \{x\}$ is not r-OSM open.
Thus X is only r-OSM open which $X - \{x\} \subseteq X$. So $smC(X - \{x\}, r) \subseteq X$.

Implies that $X - \{x\}$ is r-mgs closed, then by (2), $X - \{x\}$ is r-mb closed.

Therefore $mC(mI(X - \{x\}, r), r) \cap mI(mC(X - \{x\}, r), r) \subseteq X - \{x\}$.

Then $X - (mI(mC(\{x\}, r), r) \cup mC(mI(\{x\}, r), r))$

$= (X - mI(mC(\{x\}, r), r)) \cap (X - mC(mI(\{x\}, r), r))$

$= mC(X - mC(\{x\}, r), r) \cap mI(X - mI(\{x\}, r), r)$

$\subseteq X - \{x\}$.

Using the properties of nowhere dense of $\{x\}$, we have

$$\begin{aligned}
\{x\} &\subseteq mI(mC(\{x\}, r), r) \cup mC(mI(\{x\}, r), r) \\
&\subseteq mI(mC(\{x\}, r), r) \cup mC(mI(mC(\{x\}, r), r), r) \\
&= mI(mC(\{x\}, r), r) \cup mC(\emptyset, r) \\
&= mI(mC(\{x\}, r), r) \cup \emptyset \\
&= mI(mC(\{x\}, r), r).
\end{aligned}$$

Thus $\{x\} \subseteq mI(mC(\{x\}, r), r) = \emptyset$, contradicts.

This implies that $\{x\}$ is r-OSM closed. Therefore X is T_{gs} by Lemma 3.3.17.

(7) $\Rightarrow$ (1) Let A be r-mgb closed.

Assume that $x \in bmC(A, r)$, but $x \notin A$, then $A \subseteq X - \{x\}$.

From Lemma 3.3.17, $\{x\}$ is either r-$mpre$ open or r-OSM closed.

Case 1 : If $\{x\}$ is r-OSM closed, then $X - \{x\}$ is r-OSM open.

Since A is r-mgb closed, $bmC(A, r) \subseteq X - \{x\}$. Thus $x \notin bmC(A, r)$, contradicts.

This implies that $x \in A$, $bmC(A, r) \subseteq A \subseteq bmC(A, r)$.

Hence $bmC(A, r) = A$. Therefore by Lemma 3.2.23, A is r-mb closed.

Case 2 : If $\{x\}$ is r-$mpre$ open, then $X - \{x\}$ is r-$mpre$ closed.

Thus by Lemma 3.2.26 $pmC(A, r) \subseteq pmC(X - \{x\}, r) = X - \{x\}$.

and by Lemma 3.2.3,

$$\begin{aligned}
bmC(A, r) &= \cap\{F : F \text{ is } r - mb \text{ closed and } A \subseteq F\} \\
&\subseteq \cap\{F : F \text{ is } r - mpre \text{ closed and } A \subseteq F\} \\
&= pmC(A, r) \\
&\subseteq X - \{x\}.
\end{aligned}$$

Then $x \notin bmC(A, r)$, contradicts.

This implies that $x \in A, bmC(A,r) \subseteq A \subseteq bmC(A,r)$.

Hence $bmC(A,r) = A$. Therefore by Lemma 3.2.23, A is r-mb closed.

(3) ⇒ (7) Let $x \in X$, then by Lemma 3.3.18, $\{x\}$ is either r-$mpre$ open or nowhere dense.

Case 1 : If $\{x\}$ is r-$mpre$ open, then by Lemma 3.3.17, X is T_{gs}.

Case 2 : If $\{x\}$ is nowhere dense.

We show that $\{x\}$ is r-OSM closed.

Assume that $\{x\}$ is not r-OSM closed, then $X - \{x\}$ is not r-OSM open.

Thus X is only r-OSM open such that $X - \{x\} \subseteq X$.

Therefore $bmC(X - \{x\}, r) \subseteq X$, and so $X - \{x\}$ is r-mgb closed.

Thus by (3), $X - \{x\}$ is r-msp closed, so $mI(mC(mI(X - \{x\}, r), r), r) \subseteq X - \{x\}$.

Then

$$\begin{aligned} X - mC(mI(mC(\{x\}, r), r), r) &= mI(X - mI(mC(\{x\}, r), r), r) \\ &= mI(mC(X - mC(\{x\}, r), r), r) \\ &= mI(mC(mI(X - \{x\}, r), r), r) \\ &\subseteq X - \{x\}. \end{aligned}$$

Since $mI(mC(\{x\}, r), r) = \emptyset$, thus $\{x\} \subseteq mC(mI(mC(\{x\}, r), r), r) = mC(\emptyset, r) = \emptyset$, contradicts. This implies that, $\{x\}$ is r-OSM closed. Hence X is T_{gs} by Lemma 3.3.17.

(7) ⇒ (3) Let A be r-mgb closed.

Assume that $x \in bmC(A,r)$, but $x \notin A$, then $A \subseteq X - \{x\}$.

Thus by Lemma 3.3.17, $\{x\}$ is either r-$mpre$ open or r-OSM closed.

Case 1 : If $\{x\}$ is r-OSM closed, then $X - \{x\}$ is r-OSM open.

Since A is r-mgb closed, then $bmC(A,r) \subseteq X - \{x\}$.

Thus $x \notin bmC(A,r)$, contradicts. Implies, $x \in A$, then $bmC(A,r) \subseteq A \subseteq bmC(A,r)$, and so $bmC(A,r) = A$. Therefore by Lemma 3.2.23, A is r-mb closed.

Hence by Lemma 3.2.5, A is r-msp closed.

Case 2 : If $\{x\}$ be r-$mpre$ open, then $X - \{x\}$ is r-$mpre$ closed. Thus by Lemma 3.2.26, $pmC(A,r) \subseteq pmC(X - \{x\}) = X - \{x\}$, and by Lemma 3.2.3,

$$bmC(A,r) = \cap\{F : F \text{ is } r - mb \text{ closed and } A \subseteq F\}$$

$$\subseteq \cap\{F : F \text{ is } r-mpre \text{ closed and } A \subseteq F\}$$
$$= pmC(A, r)$$
$$\subseteq X - \{x\},$$

and so $x \notin bmC(A, r)$, contradicts. Implies, $x \in A, bmC(A, r) \subseteq A \subseteq bmC(A, r)$, so $bmC(A, r) = A$. Hence by Lemma 3.2.23, A is r-mb closed.

Therefore by Lemma 3.2.5, A is r-msp closed.

(7) ⇒ (4) Let A be r-mgp closed.

Assume that $x \in pmC(A, r)$, but $x \notin A$, then $A \subseteq X - \{x\}$.

From by Lemma 3.3.17, $\{x\}$ is either r-$mpre$ open or r-OSM closed.

Case 1: If $\{x\}$ is r-OSM closed, then $X - \{x\}$ is r-OSM open.

Since A is r-mgp closed, we get that $pmC(A, r) \subseteq X - \{x\}$.

Thus $x \notin pmC(A, r)$, contradicts. Implies, $x \in A$ and so $pmC(A, r) \subseteq A \subseteq pmC(A, r)$.

Hence $pmC(A, r) = A$. Therefore by Lemma 3.2.3, A is r-$mpre$ closed.

Case 2: If $\{x\}$ is r-$mpre$ open, then $X - \{x\}$ is r-$mpre$ closed.

Thus by Lemma 3.2.26, $pmC(A, r) \subseteq pmC(X - \{x\}, r) = X - \{x\}$,

and so $pmC(A, r) \subseteq X - \{x\}$. Therefore $x \notin pmC(A, r)$, contradicts.

Implies, $x \in A$ and so $pmC(A, r) \subseteq A \subseteq pmC(A, r)$.

Thus $pmC(A, r) = A$. Hence by Lemma 3.2.25, A is r-$mpre$ closed set.

(4) ⇒ (6) Let A be r-mgp closed, then by (4), A is r-$mpre$ closed.

Therefore by Lemma 3.2.6, A is r-msp closed.

(7) ⇒ (5) Let A be r-$mgsp$ closed.

Assume that $x \in spmC(A, r)$, but $x \notin A$, then $A \subseteq X - \{x\}$.

By Lemma 3.3.17, $\{x\}$ is either r-$mpre$ open or r-OSM closed.

Case 1: Let $\{x\}$ be r-OSM closed, then $X - \{x\}$ is r-OSM open.

Since A is r-$mgsp$ closed, $spmC(A, r) \subseteq X - \{x\}$. Thus $x \notin spmC(A, r)$, contradicts.

Implies, $x \in A$, and so $spmC(A, r) \subseteq A \subseteq spmC(A, r)$. Hence $spmC(A, r) = A$.

Therefore by Lemma 3.2.24, A is r-msp closed.

Case 2: Let $\{x\}$ be r-$mpre$ open, then $X - \{x\}$ is r-$mpre$ closed. Thus $pmC(A, r) \subseteq pmC(X - \{x\}, r) = X - \{x\}$. By Lemma 3.2.6,

$$spmC(A, r) = \cap\{F : F \text{ is } r-msp \text{ closed and } A \subseteq F\}$$

$$\subseteq \cap\{F : F \text{ is } r - mpre \text{ closed and } A \subseteq F\}$$
$$= pmC(A, r)$$
$$\subseteq X - \{x\}.$$

Therefore $x \notin spmC(A, r)$, contradicts. Implies, $x \in A$ and so $spmC(A, r) \subseteq A \subseteq spmC(A, r)$. Thus $spmC(A, r) = A$. Hence by Lemma 3.2.24, A is r-msp closed.

(5) $\Rightarrow$ (6) Let A be r-mgp closed,

then by Lemma 3.3.9, A is r-$mgsp$ closed. And by (5), A is r-msp closed.

(6) $\Rightarrow$ (7) Let $x \in X$, then by Lemma 3.3.18, $\{x\}$ is either r-$mpre$ open or nowhere dense.

Case 1 : If $\{x\}$ is r-$mpre$ open, then by Lemma 3.3.17, X is T_{gs}.

Case 2 : If $\{x\}$ is nowhere dense. Then we show that $\{x\}$ is r-OSM closed.

Assume that $\{x\}$ is not r-OSM closed, then $X - \{x\}$ is not r-OSM open.

Thus X is only r-OSM open such that $X - \{x\} \subseteq X$.

So $pmC(X - \{x\}, r) \subseteq X$, then $X - \{x\}$ is r-mgp closed.

By (6), then $X - \{x\}$ is r-msp closed.

Thus $mI(mC(mI(X - \{x\}, r), r), r) \subseteq X - \{x\}$. Then

$$\begin{aligned} X - mC(mI(mC(\{x\}, r), r), r) &= mI(X - mI(mC(\{x\}, r), r), r) \\ &= mI(mC(X - mC(\{x\}, r), r), r) \\ &= mI(mC(mI(X - \{x\}, r), r), r) \\ &\subseteq X - \{x\}. \end{aligned}$$

Hence $\{x\} \subseteq mC(mI(mC(\{x\}, r), r), r) = mC(\emptyset, r) = \emptyset$, contradicts.

So $\{x\}$ is r-OSM closed. Therefore X is T_{gs} by Lemma 3.3.17. □

Now, we give the following result.

Theorem 3.3.20 Let $(X, \mathcal{M})$ be an r-OSMS, the following statements are equivalent:

1 Evaery r-mb closed set is r-mgs closed.

2 Every r-mgb closed set is r-mgs closed.

Proof. (1) $\Rightarrow$ (2) Let A be r-mgb closed and $A \subseteq U$ where $U \in \mathcal{M}_r$,

then $bmC(A, r) \subseteq U$. Thus by Lemma 3.2.23, $bmC(A, r)$ is r-mb closed.

And so by (1), $bmC(A, r)$ is r-mgs closed. Therefore $smC(A, r) \subseteq smC(bmC(A, r), r)$ $\subseteq U$. Hence A is r-mgs closed.

(2) $\Rightarrow$ (1) Let A be r-mb closed, then by Lemma 3.3.3, A is r-mgb closed, and so by (2), A is r-mgs closed. □

Lemma 3.3.21 Let $(X, \mathcal{M})$ be an r-OSMS. If every r-mgp closed is r-msg closed, then X is T_{gs}.

Proof. Let $x \in X$, by Lemma 3.3.18, $\{x\}$ is either r-$mpre$ open set or nowhere dense.
If $\{x\}$ is r-$mpre$ open, then by Lemma 3.3.17, X is T_{gs}.
If $\{x\}$ is nowhere dense.
We show that $\{x\}$ is r-OSM closed.
Assume that $\{x\}$ is not r-OSM closed, then $X - \{x\}$ is not r-OSM open.
Thus X is only r-OSM open such that $X - \{x\} \subseteq X$, and so $pmC(X - \{x\}, r) \subseteq X$.
Implies, $X - \{x\}$ is r-mgp closed. Therefore $X - \{x\}$ is r-msg closed.
Thus $smC(X - \{x\}, r) \subseteq X - \{x\}$. By Lemma 3.2.27, we have
$(X - \{x\}) \cup mI(mC(X - \{x\}, r), r) = smC(X - \{x\}, r) \subseteq X - \{x\}$.
Thus $mI(mC(X - \{x\}, r), r) \subseteq X - \{x\}$. So, $X - \{x\}$ is r-ms closed.
This implies that $\{x\}$ is r-ms open. Then $\{x\} \subseteq mC(mI(\{x\}, r), r)$
$\subseteq mC(mI(mC(\{x\}, r), r), r) = mC(\emptyset) = \emptyset$, contradicts.
This implies that $\{x\}$ is r-OSM closed set. Hence X is T_{gs} by Lemma 3.3.17. □

Corollary 3.3.22 Let $(X, \mathcal{M})$ be an r-OSMS is r-msg closed and $\mathcal{M}$ has the property ($\mathcal{U}$), then the following statements are equivalent.

1 Every r-$mgsp$ closed set is r-$mpre$ closed.

2 Every r-mgb closed set is r-$mpre$ closed.

3 X is extremely disconnected and T_{gs}.

Proof. (1) $\Rightarrow$ (2) Let A be r-mgb closed .
Since Lemma 3.3.10, A is r-$mgsp$ closed, then by (1), A is r-$mpre$ closed.
(2) $\Rightarrow$ (3) Let A be r-mgb closed , then by (2), A is r-$mpre$ closed.
From Lemma 3.3.2, A is r-mgp closed. By Theorem 3.3.13(6), X is extremely

disconnected. Thus from A is r-$mpre$ closed and Lemma 3.2.6, A is r-msp closed. And so by Theorem 3.3.19(3), X is T_{gs}.

(3) $\Rightarrow$ (1) Let A be r-$mgsp$ closed .

Since X is extremely disconnected and by Theorem 3.3.13(1), A is r-mgp closed.

Since X is T_{gs} and by Theorem 3.3.19(4), A is r-$mpre$ closed. □

Lemma 3.3.23 If X is nowhere dense, then $X - \{x\}$ is r-ms open.

Proof. Let X be nowhere dense, then $mI(mC(\{x\}, r), r) = \emptyset$. Since $\emptyset \subseteq \{x\}$, then $mI(mC(\{x\}, r), r) \subseteq \{x\}$. Consider, $X - \{x\} \subseteq X - mI(mC(\{x\}, r), r) = mC(X - mC(\{x\}, r)), r) = mC(mI(X - \{x\}, r), r)$. Thus $X - \{x\}$ is r-ms open. □

Lemma 3.3.24 Let $(X, \mathcal{M})$ be an r-OSMS. Then every r-msg closed set is r-mb closed.

Proof. Let A be r-msg closed. Assume that $x \in bmC(A, r)$, but $x \notin A$.

Then $A \subseteq X - \{x\}$. By Lemma 3.3.18, $\{x\}$ is either r-$mpre$ open or nowhere dense.

If $\{x\}$ is r-$mpre$ open, then $X - \{x\}$ is r-$mpre$ closed. Thus by Lemma 3.2.3, $X - \{x\}$ is r-mb closed, and so $bmC(A, r) \subseteq bmC(X - \{x\}, r) = X - \{x\}$. Therefore $x \notin bmC(A, r)$, contradicts. Then $x \in A, bmC(A, r) \subseteq A \subseteq bmC(A, r)$. Thus $bmC(A, r) = A$.

Hence A is r-mb closed.

If $\{x\}$ is nowhere dense, then by Lemma 3.3.23, $X - \{x\}$ is r-ms open.

Since A is r-msg closed, $smC(A, r) \subseteq X - \{x\}$, and by Lemma 3.2.4, we have

$$\begin{aligned} bmC(A, r) &= \cap\{F : F \text{ is } r - mb \text{ closed and } A \subseteq F\} \\ &\subseteq \cap\{F : F \text{ is } r - ms \text{ closed and } A \subseteq F\} \\ &= smC(A, r) \subseteq X - \{x\}. \end{aligned}$$

Therefore $x \notin bmC(A, r)$, contradicts. Then $x \in A$, $bmC(A, r) \subseteq A \subseteq bmC(A, r)$. Thus $bmC(A, r) = A$, implies that A is r-mb closed. □

CHAPTER 4

FUZZY r-WEAKLY STRUCTURES SPACES

In this chapter, we define the fuzzy r-weakly structure spaces and introduced the concept of fuzzy r-weak α-open set, fuzzy r-weak α-semiopen set, fuzzy r-weak α-continuous and fuzzy r-weak α-open mappings it intersects on such.

4.1 Fuzzy α-$\mathcal{W}_r$ open sets

In this section, we define of open set, closed set, closure, interior, α-open set and semiopen set in fuzzy r-weak space. And some basic properties.

Definition 4.1.1 [1] Let A and B be fuzzy sets. We denote

1 $A \subseteq B \Leftrightarrow A(x) \leq B(x)$ for all $x \in X$,

2 $(\bigcup_{\alpha\in\Lambda} A_\alpha)(x) = \sup_{\alpha\in\Lambda} A_\alpha(x)$ for all $x \in X$,

3 $(\bigcap_{\alpha\in\Lambda} A_\alpha)(x) = \inf_{\alpha\in\Lambda} A_\alpha(x)$ for all $x \in X$.

Let I be the unit interval $[0, 1]$ of the real number line. A member A of I^X is called a fuzzy set of X. By $\widetilde{0}$ and $\widetilde{1}$ we donote constant maps on X with value 0 and 1, respectively. For any $A \in I^X$, A^C denotes the complement $\widetilde{1} - A$. All other notations are standard notations of fuzzy set theory.

Definition 4.1.2 Let X be a nonempty set and $r \in (0, 1]$. A fuzzy family $\mathcal{W} : I^X \to I$ on X is said to have a fuzzy r-weakly structure if the family

$$\mathcal{W}_r = \{A \in I^X : \mathcal{W}(A) \geq r\}$$

contains $\widetilde{0}$.

Then the pair $(X, \mathcal{W})$ is called a *fuzzy r-weakly structure space* (simply, r-FWS). Every member of $\mathcal{W}_r$ is called a *fuzzy r-weak open set* (simply, r-FWS open set). A fuzzy set A is called a *fuzzy r-weak closed set* (simply, r-FWS closed set) if the complement of A (simply, A^C) is a fuzzy r-weak open set.

Let $(X, \mathcal{W})$ be an r-FWS and $r \in (0, 1]$. The *fuzzy r-weak closure* of A, denote by $wC(A,r)$, is define as

$$wC(A,r) = \cap\{B \in I^X : B^C \in \mathcal{W}_r \text{ and } A \subseteq B\},$$

$$wC(A,r)(x) = \inf_{\alpha\in\Lambda}\{B_\alpha(x) \in I^X : B_\alpha^C \in \mathcal{W}_r \text{ and } A(x) \leq B(x)\} \text{ for all } x \in X.$$

The *fuzzy r-weak interior* of A, denote by $wI(A,r)$, is define as

$$wI(A,r) = \cup\{B \in I^X : B \in \mathcal{W}_r \text{ and } B \subseteq A\}$$

$$wI(A,r)(x) = \sup_{\alpha\in\Lambda}\{B_\alpha(x) \in I^X : B_\alpha \in \mathcal{W}_r \text{ and } B(x) \leq A(x)\} \text{ for all } x \in X.$$

Example 4.1.3 Let $X = [0,1]$ and let A be fuzzy sets define as follows:

$$A(x) = \begin{cases} x + \frac{1}{2}, & ; 0 \leq x \leq \frac{1}{4}, \\ \frac{1}{3}(x-1) + \frac{1}{2}, & ; \frac{1}{4} \leq x \leq 1. \end{cases}$$

Let us consider a fuzzy r-weakly structure as follows:

$$\mathcal{W}(\mu) = \begin{cases} \frac{2}{3}, & \text{if } \mu = \widetilde{0}, A, \\ 0, & \text{otherwise.} \end{cases}$$

Let $r = \frac{2}{3}$, $\mathcal{W}_{\frac{2}{3}} = \{A \in I^X : \mathcal{W}(A) \geq \frac{2}{3}\}$. $\mathcal{W}_{\frac{2}{3}} = \{\widetilde{0}, A\}$. Then
$(X, \mathcal{W})$ is $\frac{2}{3}$-FWS. Then $\widetilde{0}$ and A are $\frac{2}{3}$-FWS open and $\widetilde{1}$ and $\widetilde{1} - A$ are $\frac{2}{3}$-FWS closed.

Theorem 4.1.4 Let $(X, \mathcal{W})$ be an r-FWS and $A, B \in I^X$.

1 $wI(A,r) \subseteq A$ and if $A \in \mathcal{W}_r$, then $wI(A,r) = A$.

2 $A \subseteq wC(A,r)$ and if $A^C \in \mathcal{W}_r$, then $wC(A,r) = A$.

3 If $A \subseteq B$, then $wI(A,r) \subseteq wI(B,r)$ and $wC(A,r) \subseteq wC(B,r)$.

4 $wI(A,r) \cap wI(B,r) \supseteq wI(A \cap B, r)$ and $wC(A,r) \cup wC(B,r) \subseteq wC(A \cup B, r)$.

5 $wI(mI(A,r),r) = wI(A,r)$ and $wC(wC(A,r),r) = wC(A,r)$.

6 $\widetilde{1} - wC(A,r) = wI(\widetilde{1} - A, r)$ and $\widetilde{1} - wI(A,r) = wC(\widetilde{1} - A, r)$.

Proof. (1) Let B_α be r-FWS open such that $B_\alpha \subseteq A$ for all $\alpha \in \Lambda$.

Then, for any $x \in X$, $B_\alpha(x) \leq A(x)$ for all $\alpha \in \Lambda$.

Thus $(\bigcup_{\alpha\in\Lambda} B_\alpha)(x) = \sup_{\alpha\in\Lambda} B_\alpha(x) \leq A(x)$ for all $x \in X$.

This implies that $(wI(A,r))(x) \leq A(x)$ for all $x \in X$.

Hence $wI(A,r) \subseteq A$. Next we show that if $A \in \mathcal{W}_r$, then $wI(A,r) = A$.

Let $A \in \mathcal{W}_r$. Then $A \in \{B_\alpha \in I^X : B_\alpha \in \mathcal{W}_r$ and $B_\alpha \subseteq A$ for all $\alpha \in \Lambda\}$.

Thus for any $x \in X$, $A(x) \in \{B_\alpha(x) : B_\alpha \in \mathcal{W}_r$ and $B_\alpha \subseteq A$ for all $\alpha \in \Lambda\}$.

Thus $A(x) \leq \sup_{\alpha\in\Lambda}\{B_\alpha(x) : B_\alpha \in \mathcal{W}_r$ and $B_\alpha \subseteq A\} = (\bigcup_{\alpha\in\Lambda} B_\alpha)(x)$

and so $A \subseteq wI(A,r)$. Since $wI(A,r) \subseteq A$, we get that $wI(A,r) = A$.

(2) Let B_α be r-FWS closed such that $A \subseteq B_\alpha$ for all $\alpha \in \Lambda$.

Then, for any $x \in X$, $A(x) \leq B_\alpha(x)$ for all $\alpha \in \Lambda$.

Thus $(\bigcap_{\alpha\in\Lambda} B_\alpha)(x) = \inf_{\alpha\in\Lambda} B_\alpha(x) \geq A(x)$ for all $x \in X$.

This implies that $A(x) \leq (wC(A,r))(x)$ for all $x \in X$.

Hence $A \subseteq wC(A,r)$. Next we show that if $A^C \in \mathcal{W}_r$, then $wC(A,r) = A$.

Let $A^C \in \mathcal{W}_r$. Then $A \in \{B_\alpha \in I^X : B_\alpha^C \in \mathcal{W}_r$ and $A \subseteq B_\alpha$ for all $\alpha \in \Lambda\}$.

Thus, for any $x \in X$, $A(x) \in \{B_\alpha(x) : B_\alpha^C \in \mathcal{W}_r$ and $A \subseteq B_\alpha$ for all $\alpha \in \Lambda\}$.

Thus $A(x) \geq \inf_{\alpha\in\Lambda}\{B_\alpha(x) : B_\alpha^C \in \mathcal{W}_r$ and $A \subseteq B_\alpha\} = (\bigcap_{\alpha\in\Lambda} B_\alpha)(x)$

and so $wC(A,r) \subseteq A$. Since $A \subseteq wC(A,r)$, we have $wC(A,r) = A$.

(3) Let $A \subseteq B$, then $A(x) \leq B(x)$ for all $x \in X$.

Let $B_\beta \in \mathcal{W}_r$ such that $B_\beta \subseteq A$ for all $\beta \in \Lambda$.

Since $A \subseteq B$, we have $B_\beta \subseteq B$ for all $\beta \in \Lambda$.

Thus $B_\beta \in \{F_\alpha \in I^X : F_\alpha \subseteq B, F_\alpha \in \mathcal{W}_r$ for all $\alpha \in \Lambda\}$.

So, for any $x \in X$, $B_\beta(x) \leq \sup_{\alpha\in\Lambda}\{F_\alpha(x) : F_\alpha \in \mathcal{W}_r$ and $F_\alpha \subseteq B\}$.

Thus $\sup_{\alpha\in\Lambda}\{F_\alpha(x) : F_\alpha \in \mathcal{W}_r$ and $F_\alpha \subseteq B\}$ is an upper bound of $\{B_\beta(x) : B_\beta \in \mathcal{W}_r$ and $B_\beta \subseteq A$ for all $\beta \in \Lambda\}$. Hence $\sup_{\beta\in\Lambda}\{B_\beta(x) : B_\beta \in \mathcal{W}_r$ and $B_\beta \subseteq A\} \leq \sup_{\alpha\in\Lambda}\{F_\alpha(x) : F_\alpha \in \mathcal{W}_r$ and $F_\alpha \subseteq B\}$ for all $x \in X$. This implies that $wI(A,r) \subseteq wI(B,r)$. Next we show that $wC(A,r) \subseteq wC(B,r)$.

Let $A \subseteq B$, then $A(x) \leq B(x)$ for all $x \in X$.

Let B_β be r-FWS closed such that $A \subseteq B_\beta$ for all $\beta \in \Lambda$.

Since $A \subseteq B$, we have $B \subseteq B_\beta$ for all $\beta \in \Lambda$.

Thus $B_\beta \in \{F_\alpha \in I^X : B \subseteq F_\alpha, F_\alpha^C \in \mathcal{W}_r$ for all $\alpha \in \Lambda\}$.

So, for any $x \in X$, $B_\beta(x) \geq \inf_{\alpha\in\Lambda}\{F_\alpha(x) : F_\alpha^C \in \mathcal{W}_r$ and $F_\alpha \subseteq B\}$.
Thus $\inf_{\alpha\in\Lambda}\{F_\alpha(x) : F_\alpha^C \in \mathcal{W}_r$ and $F_\alpha \subseteq B\}$ is a lower bound of $\{B_\beta(x) : B_\beta^C \in \mathcal{W}_r$ and $A \subseteq B_\beta$ for all $\beta \in \Lambda\}$. and so $\inf_{\beta\in\Lambda}\{B_\beta(x) : B_\beta^C \in \mathcal{W}_r$ and $A \subseteq B_\beta\} \geq \inf_{\alpha\in\Lambda}\{F_\alpha(x) : F_\alpha^C \in \mathcal{W}_r$ and $F_\alpha \subseteq B\}$ for all $x \in X$. Hence $wC(A,r) \subseteq wC(B,r)$.

(4) Since $A \cap B \subseteq A$, $A \cap B \subseteq B$ and using (3), we have
$wI(A \cap B, r) \subseteq wI(A,r)$ and $wI(A \cap B, r) \subseteq wI(B,r)$.
Therefore $wI(A \cap B, r) \subseteq wI(A,r) \cap wI(B,r)$.
We show that $wC(A,r) \cup wC(B,r) \subseteq wC(A \cup B, r)$.
Since $A \subseteq A \cup B$, $B \subseteq A \cup B$ and by (3), we get that
$wC(A,r) \subseteq wC(A \cup B, r)$ and $wC(B,r) \subseteq wC(A \cup B, r)$.
Therefore $wC(A,r) \cup wC(B,r) \subseteq wC(A \cap B, r)$.

(5) By (1) and (3), we have $wI(wI(A,r),r) \subseteq wI(A,r)$.
For any $G_\beta \in I^X$ be such that $G_\beta \in \mathcal{W}_r$ and $G_\beta \subseteq A$, we have $G_\beta \subseteq wI(A,r)$.
Thus $G_\beta \subseteq \cup\{G_\alpha : G_\alpha \in \mathcal{W}_r, G_\alpha \subseteq wI(A,r)$ for all $\alpha \in \Lambda\}$. This implies that

$$\begin{aligned} wI(A,r) &= \cup\{G_\beta : G_\beta \in \mathcal{W}_r, G_\beta \subseteq A \text{ for all } \beta \in \Lambda\} \\ &\subseteq \cup\{G_\alpha : G_\alpha \in \mathcal{W}_r, G_\alpha \subseteq wI(A,r) \text{ for all } \alpha \in \Lambda\} \\ &= wI(wI(A,r),r). \end{aligned}$$

So $wI(A,r) = wI(wI(A,r),r)$. We show that $wC(wC(A,r),r) = wC(A,r)$.
It follows from (2) and (3), we get that $wC(A,r) \subseteq wC(wC(A,r),r)$.
For any $F_\beta \in I^X$ be such that $F_\beta^C \in \mathcal{W}_r$ and $A \subseteq F_\beta$, we have $wC(A,r) \subseteq F_\beta$.
Thus $\cap\{F_\alpha : F_\alpha^C \in \mathcal{W}_r, wC(A,r) \subseteq F_\alpha$ for all $\alpha \in \Lambda\} \subseteq F_\beta$. This implies that

$$\begin{aligned} wC(A,r) &= \cap\{F_\beta : F_\beta^C \in \mathcal{W}_r, A \subseteq F_\beta \text{ for all } \beta \in \Lambda\} \\ &\supseteq \cap\{F_\alpha : F_\alpha^C \in \mathcal{W}_r, wC(A,r) \subseteq F_\alpha \text{ for all } \alpha \in \Lambda\} \\ &= wC(wC(A,r),r). \end{aligned}$$

So $wC(A,r) = wC(wC(A,r),r)$.

(6) We show that $wC(\widetilde{1} - A, r) = \widetilde{1} - wI(A,r)$. For each $x \in X$ and $\alpha \in \Lambda$, we have

$$\begin{aligned} (\widetilde{1} - wI(A,r))(x) &= \widetilde{1}(x) - wI(A,r)(x) \\ &= \widetilde{1}(x) - (\cup\{G_\alpha \in I^X : G_\alpha \subseteq A, G_\alpha \in \mathcal{W}_r, \alpha \in \Lambda\})(x) \end{aligned}$$

$$
\begin{aligned}
&= \widetilde{1}(x) - \sup_{\alpha\in\Lambda}\{G_\alpha(x) : G_\alpha \subseteq A, G_\alpha \in \mathcal{W}_r\}\\
&\leq \widetilde{1}(x) - G_\alpha(x)\\
&= (\widetilde{1} - G_\alpha)(x).
\end{aligned}
$$

Thus

$$
\begin{aligned}
(\widetilde{1} - wI(A,r))(x) &\leq \inf_{\alpha\in\Lambda}\{(\widetilde{1} - G_\alpha)(x) : (\widetilde{1} - G_\alpha)^C \in \mathcal{W}_r, \widetilde{1} - A \subseteq \widetilde{1} - G_\alpha\}\\
&\leq \inf_{\alpha\in\Lambda}\{F_\alpha(x) : F_\alpha^C \in \mathcal{W}_r, \widetilde{1} - A \subseteq F_\alpha\}\\
&= (\bigcap_{\alpha\in\Lambda}\{F_\alpha : F_\alpha^C \in \mathcal{W}_r, \widetilde{1} - A \subseteq F_\alpha\})(x)\\
&= (wC(\widetilde{1} - A, r))(x).
\end{aligned}
$$

So $\widetilde{1} - wI(A,r) \subseteq wC(\widetilde{1} - A, r)$.

Consider, For any $\alpha \in \Lambda$, $G_\alpha \in \mathcal{W}_r, G_\alpha \subseteq A$, we have

$$
\begin{aligned}
G_\alpha(x) &= (\widetilde{1} - (\widetilde{1} - G_\alpha))(x)\\
&= \widetilde{1}(x) - (\widetilde{1} - G_\alpha)(x)\\
&\leq \widetilde{1}(x) - \inf_{\alpha\in\Lambda}\{(\widetilde{1} - G_\alpha)(x) : \widetilde{1} - A \subseteq \widetilde{1} - G_\alpha, (\widetilde{1} - G_\alpha)^C \in \mathcal{W}_r\}\\
&\leq \widetilde{1}(x) - \inf_{\alpha\in\Lambda}\{F_\alpha(x) : \widetilde{1} - A \subseteq F_\alpha, F_\alpha^C \in \mathcal{W}_r\}\\
&= wC(\widetilde{1} - A, r)(x)\\
&= (\widetilde{1} - wC(\widetilde{1} - A, r))(x).
\end{aligned}
$$

Thus $\widetilde{1} - wC(\widetilde{1} - A, r)(x) \geq \sup_{\alpha\in\Lambda}\{(G_\alpha)(x) : G_\alpha \subseteq A, G_\alpha \in \mathcal{W}_r\}$
$= wI(A,r)(x)$.

This implies that $wC(\widetilde{1} - A, r) \subseteq \widetilde{1} - wI(A,r)$.

Hence $wC(\widetilde{1} - A, r) = \widetilde{1} - wI(A,r)$.

Next we show that $wI(\widetilde{1} - A, r) = \widetilde{1} - wC(A,r)$.

We have

$$
\begin{aligned}
\widetilde{1} - wC(A,r) &= \widetilde{1} - wC(\widetilde{1} - (\widetilde{1} - A), r)\\
&= \widetilde{1} - (\widetilde{1} - wI(\widetilde{1} - A, r))\\
&= wI(\widetilde{1} - A, r).
\end{aligned}
$$

Hence $wI(\widetilde{1} - A, r) = \widetilde{1} - wC(A,r)$. □

Definition 4.1.5 Let $(X, \mathcal{W})$ be an r-FWS and $A \in I^X$. Then a fuzzy set A is called a *fuzzy r-weak semiopen set* (simply, r-FWS semiopen set) in X if

$$A \subseteq wC(wI(A, r), r).$$

A fuzzy set A is called a *fuzzy r-weak semiclosed set* (simply, r-FWS semiclosed set) if the complement of A is fuzzy r-weak semiopen.

Let $(X, \mathcal{W})$ and $(Y, \mathcal{N})$ be two r-FWS's. Then $f : X \to Y$ is said to be *fuzzy r-W continuous mappings* if for every $A \in \mathcal{N}_r$, $f^{-1}(A)$ is in $\mathcal{W}_r$.

Definition 4.1.6 Let $(X, \mathcal{W})$ be an r-FWS and $A \in I^X$. Then a fuzzy set A is called a *fuzzy r-weak α-open set* (simply, α-$\mathcal{W}_r$ open set) in X if

$$A \subseteq wI(wC(wI(A, r), r), r).$$

A fuzzy set A is called a *fuzzy r-weak α-closed set* (simply, α-$\mathcal{W}_r$ closed set) if the complement of A is fuzzy r-weak α-open.

Lemma 4.1.7 Let $(X, \mathcal{W})$ be an r-FWS. Then the following condition are hold:

1 Every r-FWS open set is α-$\mathcal{W}_r$ open.

2 Every α-$\mathcal{W}_r$ open set is r-FWS semiopen.

Proof. (1) Let A be r-FWS open. Then $wI(A, r) = A$.

Since $A \subseteq wC(A, r) = wC(wI(A, r), r)$, we get that

$A = wI(A, r) \subseteq wI(wC(wI(A, r), r), r)$. Hence A is α-$\mathcal{W}_r$ open.

(2) Let A be α-$\mathcal{W}_r$ open. Then $A \subseteq wI(wC(wI(A, r), r), r)$.

Since $wI(wC(wI(A, r), r), r) \subseteq wC(wI(A, r), r)$,

we get that $A \subseteq wC(wI(A, r), r)$. Hence A is r-FWS semiopen. $\square$

Remark 4.1.8 The following implications are obtained but the converses are not trure in general.

fuzzy r-weak open $\Rightarrow$ fuzzy r-weak α-open $\Rightarrow$ fuzzy r-weak semiopen

Example 4.1.9 Let $X = [0, 1]$ and let A and B be fuzzy sets define as follows

$$A(x) = \begin{cases} x + \frac{1}{2}, & ; 0 \leq x \leq \frac{1}{4}, \\ \frac{1}{3}(x-1) + \frac{1}{2}, & ; \frac{1}{4} < x \leq 1, \end{cases}$$

and $B(x) = \frac{1}{3}; 0 \le x \le 1.$

Let us consider a fuzzy r-weak structure define as follows

$$\mathcal{W}(\mu) = \begin{cases} \frac{1}{2}, & \text{if } \mu = \widetilde{0}, A, \\ 0, & \text{otherwise.} \end{cases}$$

Let $r = \frac{1}{2}$, consider $wI(wC(wI(B, \frac{1}{2}), \frac{1}{2}), \frac{1}{2})(x) = (\frac{1}{2}, \frac{3}{4}]$.

Then $B(x) \le wI(wC(wI(B, \frac{1}{2}), \frac{1}{2}), \frac{1}{2})(x)$, Thus $B \subseteq wI(wC(wI(B, \frac{1}{2}), \frac{1}{2}), \frac{1}{2})$.

Therefore B is α-$\mathcal{W}_{\frac{1}{2}}$ open. But B is not $\frac{1}{2}$-FWS open.

Let $r = \frac{1}{2}$, consider $wC(wI(\widetilde{1} - B, \frac{1}{2}), \frac{1}{2})(x) = [\frac{2}{3}, \frac{3}{4})$.

Then $(\widetilde{1} - B)(x) \le wC(wI((\widetilde{1} - B, \frac{1}{2}), \frac{1}{2})(x)$, Thus $(\widetilde{1} - B) \subseteq wC(wI((\widetilde{1} - B), \frac{1}{2}), \frac{1}{2})$.

Therefore $(\widetilde{1} - B)$ is $\frac{1}{2}$-weak semiopen. But $(\widetilde{1} - B)$ is not α-$\mathcal{W}_{\frac{1}{2}}$ open.

Lemma 4.1.10 Let $(X, \mathcal{W})$ be an r-FWS. Then a fuzzy set A is α-$\mathcal{W}_r$ closed set if and only if $wC(wI(wC(A, r), r), r) \subseteq A$.

Proof. ($\Rightarrow$) Let A be α-$\mathcal{W}_r$ closed. Then $\widetilde{1} - A$ is α-$\mathcal{W}_r$ open.

Thus $\widetilde{1} - A \subseteq wI(wC(wI(\widetilde{1} - A, r), r), r)$. By Theorem 4.1.4 (6), we have

$$\begin{aligned} \widetilde{1} - A &\subseteq wI(wC(\widetilde{1} - wC(A, r), r), r) \\ &= wI(\widetilde{1} - wI(wC(A, r), r), r) \\ &= \widetilde{1} - wC(wI(wC(A, r), r), r). \end{aligned}$$

Hence $wC(wI(wC(A, r), r), r) \subseteq A$.

($\Leftarrow$) Let $wC(wI(wC(A, r), r), r) \subseteq A$. Consider,

$$\begin{aligned} \widetilde{1} - A &\subseteq \widetilde{1} - wC(wI(wC(A, r), r), r) \\ &= wI(\widetilde{1} - wI(wC(A, r), r), r) \\ &= wI(wC(\widetilde{1} - wC(A, r), r), r) \\ &= wI(wC(wI(\widetilde{1} - A, r), r), r). \end{aligned}$$

Thus $\widetilde{1} - A$ is α-$\mathcal{W}_r$ open. This implies that A is α-$\mathcal{W}_r$ closed. □

Theorem 4.1.11 Let $(X, \mathcal{W})$ be an r-FWS. Then any union of α-$\mathcal{W}_r$ open set is α-$\mathcal{W}_r$ open.

Proof. Let A_β be α-$\mathcal{W}_r$ open for all $\beta \in \Lambda$. Then for any $\beta \in \Lambda$,
$A_\beta \subseteq wI(wC(wI(A_\beta, r), r), r) \subseteq wI(wC(wI(\bigcup_{\beta\in\Lambda} A_\beta, r), r), r)$.
Since $A_\beta(x) \leq wI(wC(wI(\bigcup_{\beta\in\Lambda} A_\beta, r), r), r)(x)$ for all $x \in X$,
and so $\bigcup_{\beta\in\Lambda} A_\beta(x) \leq wI(wC(wI(\bigcup_{\beta\in\Lambda} A_\beta, r), r), r)(x)$ for all $x \in X$.
Thus $\bigcup_{\beta\in\Lambda} A_\beta \subseteq wI(wC(wI(\bigcup_{\beta\in\Lambda} A_\beta, r), r), r)$. Hence $\bigcup_{\beta\in\Lambda} A_\beta$ is α-$\mathcal{W}_r$ open. □

Definition 4.1.12 Let $(X, \mathcal{W})$ be an r-FWS. For any $A \in I^X$, $w\alpha C(A, r)$ and $w\alpha I(A, r)$, respectively, are define as the following:

$$w\alpha C(A, r) = \cap\{F \in I^X : A \subseteq F, F \text{ is } \alpha\text{-}\mathcal{W}_r \text{ closed }\};$$

$$w\alpha I(A, r) = \cup\{U \in I^X : U \subseteq A, U \text{ is } \alpha\text{-}\mathcal{W}_r \text{ open }\}.$$

Theorem 4.1.13 Let $(X, \mathcal{W})$ be an r-FWS and $A \in I^X$. Then the following statements are hold.

1 $w\alpha I(A, r) \subseteq A$.

2 If $A \subseteq B$, then $w\alpha I(A, r) \subseteq w\alpha I(B, r)$.

3 A is α-$\mathcal{W}_r$ open if and only if $w\alpha I(A, r) = A$.

4 $w\alpha I(w\alpha I(A, r), r) = w\alpha I(A, r)$.

5 $w\alpha C(\widetilde{1} - A, r) = \widetilde{1} - w\alpha I(A, r)$ and $w\alpha I(\widetilde{1} - A, r) = \widetilde{1} - w\alpha C(A, r)$.

Proof. (1) Let B_β be α-$\mathcal{W}_r$ open such that $B_\beta \subseteq A$ for all $\beta \in \Lambda$.
Then, for any $x \in X$, $B_\beta(x) \leq A(x)$ for all $\beta \in \Lambda$.
Thus $(\bigcup_{\beta\in\Lambda} B_\beta)(x) = \sup_{\beta\in\Lambda} B_\beta(x) \leq A(x)$ for all $x \in X$.
This implies that $(w\alpha I(A, r))(x) \leq A(x)$ for all $x \in X$.
Hence $w\alpha I(A, r) \subseteq A$.

(2) Let $A \subseteq B$. Then $A(x) \leq B(x)$ for all $x \in X$.
Let B_β be α-$\mathcal{W}_r$ open such that $B_\beta \subseteq A$ for all $\beta \in \Lambda$.
Since $A \subseteq B$, we have $B_\beta \subseteq B$ for all $\beta \in \Lambda$.
Thus $B_\beta \in \{F_\alpha \in I^X : F_\alpha \subseteq B, F_\alpha \text{ is } \alpha\text{-}\mathcal{W}_r \text{ open}\}$.

So, for any $x \in X$, $B_\beta(x) \leq \sup_{\alpha\in\Lambda}\{F_\alpha(x) : F_\alpha$ is α-$\mathcal{W}_r$ open and $F_\alpha \subseteq B$ for all $\alpha \in \Lambda\}$. Thus $\sup_{\alpha\in\Lambda}\{F_\alpha(x) : F_\alpha$ is α-$\mathcal{W}_r$ open and $F_\alpha \subseteq B$ for all $\alpha \in \Lambda\}$ is upper bound of $\{B_\beta(x) : B_\beta$ is α-$\mathcal{W}_r$ open and $B_\beta \subseteq A$ for all $\beta \in \Lambda\}$. Hence $\sup_{\beta\in\Lambda}\{B_\beta(x) : B_\beta$ is α-$\mathcal{W}_r$ open and $B_\beta \subseteq A$ for all $\beta \in \Lambda\} \leq \sup_{\alpha\in\Lambda}\{F_\alpha(x) : F_\alpha$ is α-$\mathcal{W}_r$ open and $F_\alpha \subseteq B$ for all $\alpha \in \Lambda\}$ for all $x \in X$.

This implies that $w\alpha I(A, r) \subseteq w\alpha I(B, r)$.

(3) ($\Rightarrow$) Let A be α-$\mathcal{W}_r$ open.

Then $A \in \{B_\beta \in I^X : B_\beta$ is α-$\mathcal{W}_r$ open and $B_\beta \subseteq A$ for all $\beta \in \Lambda\}$.

Thus, for any $x \in X$, $A(x) \in \{B_\beta(x) : B_\beta$ is α-$\mathcal{W}_r$ open and $B_\beta \subseteq A$ for all $\beta \in \Lambda\}$.

So $A(x) \leq \sup_{\beta\in\Lambda}\{B_\beta(x) : B_\beta$ is α-$\mathcal{W}_r$ open and $B_\beta \subseteq A$ for all $\beta \in \Lambda\} = w\alpha I(A, r)(x)$.

By (1), we have $w\alpha I(A, r) = A$.

($\Leftarrow$) Let $w\alpha I(A, r) = A$. By Theorem 4.1.11, we have $w\alpha I(A, r)$ is α-$\mathcal{W}_r$ open.

Therefore A is α-$\mathcal{W}_r$ open.

(4) By Theorem 4.1.11, we have $w\alpha I(A, r)$ is α-$\mathcal{W}_r$ open.

From (3), $w\alpha I(w\alpha I(A, r), r) = w\alpha I(A, r)$.

(5) We show that $w\alpha C(\widetilde{1} - A, r) = \widetilde{1} - w\alpha I(A, r)$.

For any $\alpha \in \Lambda$, let G_α is α-$\mathcal{W}_r$ open and $G_\alpha \subseteq A$ and $x \in X$,

$$\begin{aligned}
(\widetilde{1} - w\alpha I(A, r))(x) &= \widetilde{1}(x) - w\alpha I(A, r)(x) \\
&= \widetilde{1}(x) - \sup_{\alpha\in\Lambda}\{G_\alpha(x) : G_\alpha \subseteq A, G_\alpha \text{ is} \alpha - \mathcal{W}_r \text{ open}\} \\
&\leq \widetilde{1}(x) - G_\alpha(x) \\
&= (\widetilde{1} - G_\alpha)(x).
\end{aligned}$$

Thus

$$\begin{aligned}
(\widetilde{1} - w\alpha I(A, r))(x) &\leq \inf_{\alpha\in\Lambda}\{(\widetilde{1} - G_\alpha)(x) : (\widetilde{1} - G_\alpha) \text{ is } \alpha - \mathcal{W}_r \text{ closed}, \widetilde{1} - A \subseteq \widetilde{1} - G_\alpha\} \\
&= \inf_{\alpha\in\Lambda}\{F_\alpha(x) : F \text{ is } \alpha - \mathcal{W}_r \text{ closed}, \widetilde{1} - A \subseteq F_\alpha\} \\
&= \Big(\bigcap_{\alpha\in\Lambda}\{F_\alpha : F \text{ is } \alpha - \mathcal{W}_r \text{ closed}, \widetilde{1} - A \subseteq F_\alpha\}\Big)(x) \\
&= (w\alpha C(\widetilde{1} - A, r))(x).
\end{aligned}$$

So $\widetilde{1} - w\alpha I(A,r) \subseteq w\alpha C(\widetilde{1} - A, r)$.

Consider, for any $\alpha \in \Lambda$, $G_\alpha \in I^X, G_\alpha$ is α-$\mathcal{W}_r$ open, $G_\alpha \subseteq A$. Then

$$\begin{aligned}
G_\alpha(x) &= (\widetilde{1} - (\widetilde{1} - G_\alpha))(x) \\
&= \widetilde{1}(x) - (\widetilde{1} - G_\alpha)(x) \\
&\leq \widetilde{1}(x) - \inf_{\alpha \in \Lambda}\{(F_\alpha)(x) : \widetilde{1} - A \subseteq F_\alpha, F_\alpha^C \text{ is } \alpha - \mathcal{W}_r \text{ closed}\} \\
&= \widetilde{1}(x) - w\alpha C(\widetilde{1} - A, r)(x) \\
&= (\widetilde{1} - w\alpha C(\widetilde{1} - A, r))(x).
\end{aligned}$$

Thus

$w\alpha I(A,r)(x) = \sup_{\alpha \in \Lambda}\{(G_\alpha)(x) : G_\alpha$ is α-$\mathcal{W}_r$ open and $G_\alpha \subseteq A\} \leq (\widetilde{1} - w\alpha C(\widetilde{1} - A, r))(x)$ for all $x \in X$. Hence $w\alpha C(\widetilde{1} - A, r) \subseteq \widetilde{1} - w\alpha I(A,r)$.

This implies that $w\alpha C(\widetilde{1} - A, r) = \widetilde{1} - w\alpha I(A,r)$.

Next we show that $w\alpha I(\widetilde{1} - A, r) = \widetilde{1} - w\alpha C(A,r)$.

We have

$$\begin{aligned}
\widetilde{1} - w\alpha C(A,r) &= \widetilde{1} - w\alpha C(\widetilde{1} - (\widetilde{1} - A), r) \\
&= \widetilde{1} - (\widetilde{1} - w\alpha I(\widetilde{1} - A, r)) \\
&= w\alpha I(\widetilde{1} - A, r).
\end{aligned}$$

Hence $w\alpha I(\widetilde{1} - A, r) = \widetilde{1} - w\alpha C(A,r)$. □

Theorem 4.1.14 Let $(X, \mathcal{W})$ be an r-FWS and $A \in I^X$. Then

1 $A \subseteq w\alpha C(A,r)$.

2 If $A \subseteq B$, then $w\alpha C(A,r) \subseteq w\alpha C(B,r)$,

3 A is α-$\mathcal{W}_r$ closed if and only if $w\alpha C(A,r) = A$,

4 $w\alpha C(w\alpha I(A,r), r) = w\alpha C(A,r)$.

Proof. (1) Since
$$\begin{aligned}
A &= \widetilde{1} - (\widetilde{1} - A) \\
&\subseteq \widetilde{1} - w\alpha I(\widetilde{1} - A, r) \\
&= w\alpha C(\widetilde{1} - (\widetilde{1} - A), r) \\
&= w\alpha C(A,r).
\end{aligned}$$

Hence $A \subseteq w\alpha C(A,r)$.

(2) Let $A \subseteq B$. Then $\widetilde{1} - B \subseteq \widetilde{1} - A$.
By Theorem 4.1.13 (2), then $w\alpha I(\widetilde{1} - B, r) \subseteq w\alpha I(\widetilde{1} - A, r)$.
Thus by Theorem 4.1.13 (5), $\widetilde{1} - w\alpha C(B, r) \subseteq \widetilde{1} - w\alpha(A, r)$.
Hence $w\alpha C(A, r) \subseteq w\alpha C(B, r)$.

(3) ($\Rightarrow$) Let A be α-$\mathcal{W}_r$ closed. Then $\widetilde{1} - A$ is α-$\mathcal{W}_r$ open.
Thus by Theorem 4.1.13 (3) , $w\alpha I(\widetilde{1} - A, r) = \widetilde{1} - A$.
By Theorem 4.1.13 (5), then $\widetilde{1} - w\alpha C(A, r) = \widetilde{1} - A$.
This implies that $w\alpha C(A, r) = A$.
($\Leftarrow$) Let $w\alpha C(A, r) = A$. By Theorem 4.1.13 (5),
$\widetilde{1} - w\alpha C(A, r) = w\alpha I(\widetilde{1} - A, r)$. Then by Theorem 4.1.11, $w\alpha I(\widetilde{1} - A, r)$ is α-$\mathcal{W}_r$ open. Therefore $\widetilde{1} - w\alpha C(A, r)$ is α-$\mathcal{W}_r$ open. Thus $w\alpha C(A, r)$ is α-$\mathcal{W}_r$ closed. Hence A is α-$\mathcal{W}_r$ closed.

(4) Since $w\alpha C(A, r)$ is α-$\mathcal{W}_r$ closed. By (3), $w\alpha C(w\alpha C(A, r), r) = w\alpha C(A, r)$. □

4.2 Fuzzy r-W α-continuity and fuzzy r-W α-open mappings

In this section, we introduce the concept of fuzzy α-continuous, fuzzy α-semicontinuous and fuzzy α-open mappings in fuzzy r-weak spaces. And the relationship between the fuzzy α-open mappings and the fuzzy α-continues.

Definition 4.2.1 Let $f : (X, \mathcal{W}) \to (Y, \mathcal{N})$ be a mapping on two r-FWS's. Then f is said to be *fuzzy r-W α-continuous* if $f^{-1}(U)$ is α-$\mathcal{W}_r$ open for all r-FWS open set U in Y.

Definition 4.2.2 Let $f : (X, \mathcal{W}) \to (Y, \mathcal{N})$ be a mapping on two r-FWS's. Then f is said to be *fuzzy r-W semicontinuous* if $f^{-1}(U)$ is r-FWS semiopen for all r-FWS open U in Y.

Lemma 4.2.3 Let $(X, \mathcal{W})$ be an r-FWS. Then the following condition are hold.

1 Every fuzzy r-W continuous is fuzzy r-W α-continuous.

2 Every fuzzy r-W α-continuous is fuzzy r-W semicontinuous.

Proof. (1) Let f be a fuzzy r-W continuous, and U be r-FWS open in Y.

Thus $f^{-1}(U)$ is r-FWS open in X. By Lemma 4.1.7 (1),

then $f^{-1}(U)$ is α-$\mathcal{W}_r$ open. Hence f is fuzzy r-W α-continuous.

(2) Let f be a fuzzy r-W α-continuous, and U be r-FWS open in Y.

Thus $f^{-1}(U)$ is α-$\mathcal{W}_r$ open in X. By Lemma 4.1.7 (2),

then $f^{-1}(U)$ is r-FWS semiopen. Hence f is fuzzy r-W semicontinuous. □

Remark 4.2.4 It is obvious that every fuzzy r-W α-continuous mapping is fuzzy r-W semicontinuous but the converse may not be true as show in the next example.

fuzzy r-W continuous $\Rightarrow$ fuzzy r-W α-continuous $\Rightarrow$ fuzzy r-W semicontinuous.

Example 4.2.5 Let $X = [0, 1]$ and let A, B and C be fuzzy sets define as follows

$$A(x) = \begin{cases} x + \frac{1}{2}, & ; 0 \leq x \leq \frac{1}{4}, \\ \frac{1}{3}(x-1) + \frac{1}{2}, & ; \frac{1}{4} < x \leq 1, \end{cases}$$

$B(x) = \frac{1}{3}(x + 2); 0 \leq x \leq 1$, and

$C(x) = \frac{1}{2}x; 0 \leq x \leq 1$.

Let us consider a fuzzy r-weak structure define as follows

$$\mathcal{W}(\mu) = \begin{cases} \frac{3}{4}, & \text{if } \mu = \widetilde{0}, A, \\ 0, & \text{otherwise}, \end{cases}$$

$$\mathcal{N}(\mu) = \begin{cases} \frac{3}{4}, & \text{if } \mu = \widetilde{0}, B, \\ 0, & \text{otherwise}, \end{cases}$$

Let $r = \frac{3}{4}$, we have $\mathcal{W}_{\frac{3}{4}} = \{\widetilde{0}, A\}$ and $N_{\frac{3}{4}} = \{\widetilde{0}, B\}$.

Let $f : (X, \mathcal{W}) \to (X, \mathcal{N})$ be an identity function.

1. We show that f is fuzzy r-W α-continuous.

   Let $U \in \mathcal{N}_{\frac{3}{4}} = \{\widetilde{0}, B\}$, then $f^{-1}(U)$ is α-$\mathcal{W}_{\frac{3}{4}}$ open. Consider,

   If $U = \widetilde{0}$, $f^{-1}(U) = \widetilde{0}$ such that $\widetilde{0}(x) \leq wI(wC(wI(\widetilde{0}, \frac{3}{4}), \frac{3}{4}), \frac{3}{4})(x)$.

   So $\widetilde{0} \subseteq wI(wC(wI(\widetilde{0}, \frac{3}{4}), \frac{3}{4}), \frac{3}{4})$. Therefore $\widetilde{0}$ is α-$\mathcal{W}_{\frac{3}{4}}$ open.

   If $U = B$, $f^{-1}(U) = B$ such that $B(x) = [\frac{2}{3}, 1] \leq [\frac{2}{3}, 1] = wI(wC(wI(B, \frac{3}{4}), \frac{3}{4}), \frac{3}{4})(x)$.

So $\subseteq wI(wC(wI(B,\frac{3}{4}),\frac{3}{4}),\frac{3}{4})$. Therefore B is α-$\mathcal{W}_{\frac{3}{4}}$ open.

Hence f is fuzzy r-W α-continuous. But f is not fuzzy r-W continuous.

Let $r=\frac{3}{4}$, we have $\mathcal{W}_{\frac{3}{4}}=\{\widetilde{0},A\}$ and $N_{\frac{3}{4}}=\{\widetilde{0},B\}$.

Let $f:(X,\mathcal{N})\rightarrow(X,\mathcal{W})$ be an identity function.

2. We show that f is fuzzy r-W semicontinuous.

   Let $U\in\mathcal{W}_{\frac{3}{4}}=\{\widetilde{0},A\}$, then $f^{-1}(U)$ is $\frac{3}{4}$-FWS semiopen. Consider,

   If $U=\widetilde{0}$, $f^{-1}(U)=\widetilde{0}$ such that $\widetilde{0}(x)\leq wC(wI(\widetilde{0},\frac{3}{4}),\frac{3}{4})(x)$.

   So $\widetilde{0}\subseteq wC(wI(\widetilde{0},\frac{3}{4}),\frac{3}{4})$. Therefore $\widetilde{0}$ is $\frac{3}{4}$-FWS semiopen.

   If $U=A$, $f^{-1}(U)=A$ such that $A(x)=[\frac{2}{3},1]\leq[\frac{2}{3},1]=wC(wI(A,\frac{3}{4}),\frac{3}{4})$.

   So $A\subseteq wC(wI(A,\frac{3}{4}),\frac{3}{4})$. Therefore A is $\frac{3}{4}$-FWS semiopen.

   Hence f is fuzzy r-W semicontinuous. But f is not fuzzy$r$-W α-continuous.

consider $wI(wC(wI(B,\frac{1}{2}),\frac{1}{2}),\frac{1}{2})(x)=(\frac{1}{2},\frac{3}{4}]$.
Then $B(x)\leq wI(wC(wI(B,\frac{1}{2}),\frac{1}{2}),\frac{1}{2})(x)$, Thus $B\subseteq wI(wC(wI(B,\frac{1}{2}),\frac{1}{2}),\frac{1}{2})$.
Therefore B is α-$\mathcal{W}_{\frac{1}{2}}$ open. But B is not $\frac{1}{2}$-FWS open.
Let $r=\frac{1}{2}$, consider $wC(wI(\widetilde{1}-B,\frac{1}{2}),\frac{1}{2})(x)=[\frac{2}{3},\frac{3}{4})$.
Then $(\widetilde{1}-B)(x)\leq wC(wI((\widetilde{1}-B,\frac{1}{2}),\frac{1}{2})(x)$, Thus $(\widetilde{1}-B)\subseteq wC(wI((\widetilde{1}-B),\frac{1}{2}),\frac{1}{2})$.
Therefore $(\widetilde{1}-B)$ is $\frac{1}{2}$-weak semiopen. But $(\widetilde{1}-B)$ is not α-$\mathcal{W}_{\frac{1}{2}}$ open.

Theorem 4.2.6 Let $f:(X,\mathcal{W})\rightarrow(Y,\mathcal{N})$ be a mapping on two r-FWS's. Then the following statements are equivalent:

1 f is fuzzy r-W α-continuous,

2 $f^{-1}(B)$ is α-$\mathcal{W}_r$ closed for each r-FWS closed set B in Y,

3 $f(w\alpha C(A,r))\subseteq wC(f(A),r)$ for $A\in I^X$,

4 $w\alpha C(f^{-1}(B),r)\subseteq f^{-1}(wC(B,r))$ for $B\in I^Y$,

5 $f^{-1}(wI(B,r))\subseteq w\alpha I(f^{-1}(B),r)$ for $B\in I^Y$.

Proof. (1) ⇒ (2) Let B be r-FWS closed in Y, then $\widetilde{1}-B$ is r-FWS open in Y.
By (1), then $f^{-1}(\widetilde{1}-B)$ is α-$\mathcal{W}_r$ open in X. Hence $f^{-1}(B)$ is α-$\mathcal{W}_r$ closed.
(2) ⇒ (3) For $A\in I^X$, we have
$f^{-1}(wC(f(A),r))=f^{-1}(\bigcap_{\alpha\in\Lambda}\{F_\alpha\in I^Y:f(A)\subseteq F_\alpha$ and F_α is r - FWS closed$\}$

$$= \bigcap_{\alpha\in\Lambda}\{f^{-1}(F_\alpha) \in I^X : A \subseteq f^{-1}(F_\alpha) \text{ and } f^{-1}(F_\alpha) \text{ is } \alpha-\mathcal{W}_r \text{ closed}\}$$
$$\supseteq \bigcap_{\alpha\in\Lambda}\{K \in I^X : A \subseteq K \text{ and } K \text{ is } \alpha-\mathcal{W}_r \text{ closed}\}$$
$$= w\alpha C(A, r).$$

Then $w\alpha C(A, r) \subseteq f^{-1}(wC(f(A), r))$.

Therefore $f(w\alpha C(A, r)) \subseteq f(f^{-1}(wC(f(A), r))) \subseteq wC(f(A), r)$.

Hence $f(w\alpha C(A, r)) \subseteq wC(f(A), r)$.

(3) $\Rightarrow$ (4) For $B \in I^Y$. Then

$f(w\alpha C(f^{-1}(B), r)) \subseteq wC(f(f^{-1}(B)), r) \subseteq wC(B, r)$.

Thus $f(w\alpha C(f^{-1}(B), r)) \subseteq wC(B, r)$.

Therefore $w\alpha C(f^{-1}(B), r) \subseteq f^{-1}(f(w\alpha C(f^{-1}(B), r))) \subseteq f^{-1}(wC(B, r))$.

Hence $w\alpha C(f^{-1}(B), r) \subseteq f^{-1}(wC(B, r))$.

(4) $\Rightarrow$ (5) For $B \in I^Y$, by Theorem 4.1.13 (5), we have

$$\begin{aligned} f^{-1}(wI(B, r)) &= f^{-1}(\widetilde{1}_Y - wC(\widetilde{1}_Y - B, r)) \\ &= \widetilde{1}_X - f^{-1}(wC(\widetilde{1}_Y - B, r)) \\ &\subseteq \widetilde{1}_X - w\alpha C(f^{-1}(\widetilde{1}_Y - B, r)) \\ &= w\alpha I(f^{-1}(B), r). \end{aligned}$$

Therefore $f^{-1}(wI(B, r)) \subseteq w\alpha I(f^{-1}(B), r)$.

(5) $\Rightarrow$ (1) For $B \in \mathcal{N}_r$. Thus $f^{-1}(wI(B, r)) \subseteq w\alpha I(f^{-1}(B), r)$.

Therefore $f^{-1}(B) = f^{-1}(wI(B, r)) \subseteq w\alpha I(f^{-1}(B), r)$.

By Theorem 4.1.13 (1), $w\alpha I(f^{-1}(B), r) \subseteq f^{-1}(B)$. Thus $f^{-1}(B) = w\alpha I(f^{-1}(B), r)$.

Therefore by Theorem 4.1.11, $w\alpha I(f^{-1}(B), r)$ is α-$\mathcal{W}_r$ open.

Hence f is fuzzy r-W α-continuous. □

Definition 4.2.7 Let $f : (X, \mathcal{W}) \to (Y, \mathcal{N})$ be a mapping on two r-FWS's. Then

1 f is said to be *fuzzy r-W α-open* if for r-FWS open set A in X, $f(A)$ is α-$\mathcal{W}_r$ open in Y;

2 f is said to be *fuzzy r-W α-closed* if for r-FWS closed set A in X, $f(A)$ is

α-$\mathcal{W}_r$ closed in Y.

Theorem 4.2.8 Let $f : (X, \mathcal{W}) \to (Y, \mathcal{N})$ be a mapping on two r-FWS's. Then the following conditions are equivalent:

1 f is fuzzy r-W α-open,

2 $f(wI(A, r)) \subseteq w\alpha I(f(A), r)$ for $A \in I^X$,

3 $wI(f^{-1}(B), r) \subseteq f^{-1}(w\alpha I(B, r))$ for $B \in I^Y$.

Proof. (1) $\Rightarrow$ (2) For $A \in I^X$,

$$\begin{aligned} f(wI(A, r) &= f(\cup\{B \in I^X : B \subseteq A, B \text{ is } r \text{ - FWS open}\}) \\ &= \cup\{f(B) \in I^X : f(B) \subseteq f(A), f(B) \text{ is } \alpha - \mathcal{W}_r \text{ open}\} \\ &\subseteq \cup\{U \in I^X : U \subseteq f(A), U \text{ is } \alpha - \mathcal{W}_r \text{ open}\} \\ &= w\alpha I(f(A), r). \end{aligned}$$

Hence $f(wI(A, r) \subseteq w\alpha I(f(A), r)$.

(2) $\Rightarrow$ (1) Let A be r-FWS open in X, then $A = wI(A, r)$.

By (2), $f(A) \subseteq w\alpha I(f(A), r)$. Thus $f(A)$ is α-$\mathcal{W}_r$ open.

Hence f is fuzzy r-W α-open.

(2) $\Rightarrow$ (3) For $B \in I^Y$, it follows from (2) that

$f(wI(f^{-1}(B), r)) \subseteq w\alpha I(f(f^{-1}(B)), r) \subseteq w\alpha I(B, r)$.

Thus $f(wI(f^{-1}(B), r)) \subseteq w\alpha I(B, r)$.

Therefore $wI(f^{-1}(B), r) \subseteq f^{-1}(f(wI(f^{-1}(B), r))) \subseteq f^{-1}(w\alpha I(B, r))$.

Hence $wI(f^{-1}(B), r) \subseteq f^{-1}(w\alpha I(B, r))$.

(3) $\Rightarrow$ (2) For $A \in I^X$,

then $wI(f^{-1}(f(A)), r) \subseteq f^{-1}(w\alpha I(f(A), r)$. Since $wI(A, r) \subseteq wI(f^{-1}(f(A)), r)$.

Thus $wI(A, r) \subseteq f^{-1}(w\alpha I(f(A), r))$. Hence $f(wI(A, r)) \subseteq w\alpha I(f(A), r)$. □

Theorem 4.2.9 Let $f : (X, \mathcal{W}) \to (Y, \mathcal{N})$ be a mapping on two r-FWS's. Then the following statements are equivalent:

1 f is fuzzy r-W α-closed.

2 $w\alpha C(f(A), r)) \subseteq f(wC(A, r))$ for $A \in I^X$.

Proof. (1) ⇒ (2) For $A \in I^X$, Then

$$f(wC(A,r)) = f(\cap\{F \in I^X : A \subseteq F, F \text{ is } r \text{ - FWS open}\})$$
$$= \cap\{f(F) \in I^Y : f(A) \subseteq f(F), f(F) \text{ is } \alpha - \mathcal{W}_r \text{ open}\}$$
$$\supseteq \cap\{K \in I^Y : f(A) \subseteq K, K \text{ is } \alpha - \mathcal{W}_r \text{ open}\}$$
$$= w\alpha C(f(A), r).$$

Therefore $w\alpha C(f(A), r) \subseteq f(wC(A,r))$.

(2) ⇒ (1) Let A be r-FWS closed in X,

then $A = wC(A,r)$. By (2), $w\alpha C(f(A), r)) \subseteq f(wC(A,r)) = f(A)$.

Thus $f(A)$ is α-$\mathcal{W}_r$ closed. Hence f is fuzzy r-W α-closed. □

Definition 4.2.10 Let $f : (X, \mathcal{W}) \to (Y, \mathcal{N})$ be a mapping on two r-FWS's. Then

1 f is said to be *fuzzy r-$W^*\alpha$-open* if for every α-$\mathcal{W}_r$ open A in X, $f(A)$ is r-FWS open in Y;

2 f is said to be *fuzzy r-$W^*\alpha$-closed* if for every α-$\mathcal{W}_r$ closed A in X, $f(A)$ is r-FWS closed in Y.

Definition 4.2.11 Let X be a nonempty set and $\mathcal{W} : I^X \to I$ a fuzzy family on X. The fuzzy r-weak structure $\mathcal{W}$ has the property $(\mathcal{U})$ if for $A_i \in \mathcal{W}_r (i \in J)$,

$$\mathcal{W}(\cup A_i) \geq \wedge \mathcal{W}(A_i).$$

Theorem 4.2.12 Let $(X, \mathcal{W})$ be an r-FMS such that $\mathcal{W}$ has the property $(\mathcal{U})$. Then for $A \in I^X$, $wI(A,r) = A$ if and only if A is r-FWS open.

Proof. Let for $A \in I^X$, then $wI(A,r) = A$. By property $(\mathcal{U})$, then a r-FWS open set A. Conversely, for $A \in I^X$, From Theorem 4.1.4(1), then $wI(A,r) = A$. □

Theorem 4.2.13 Let $f : (X, \mathcal{W}) \to (Y, \mathcal{N})$ be a mapping on two r-FWS's. Then the following statements are equivalent:

1 $f(w\alpha I(A,r)) \subseteq wI(f(A), r)$ for $A \in I^X$,

2 $w\alpha I(f^{-1}(B), r) \subseteq f^{-1}(wI(B,r))$ for $B \in I^Y$.

Proof. (1) ⇒ (2) For $B \in I^Y$,

By (1), then $f(w\alpha I(f^{-1}(B), r)) \subseteq wI(f(f^{-1}(B)), r) \subseteq wI(B, r)$.

Thus $f(w\alpha I(f^{-1}(B), r)) \subseteq wI(B, r)$.

Therefore $w\alpha I(f^{-1}(B), r) \subseteq f^{-1}f(w\alpha I(f^{-1}(B), r)) \subseteq f^{-1}(wI(B, r))$.

Hence $w\alpha I(f^{-1}(B), r) \subseteq f^{-1}(wI(B, r))$.

(2) ⇒ (1) For $A \in I^X$, then $w\alpha I(f^{-1}(f(A)), r) \subseteq f^{-1}(wI(f(A), r)$.

Since $f(w\alpha I(A, r)) \subseteq f(wI(f^{-1}(f(A)), r)) \subseteq f(f^{-1}(wI(f(A), r))) \subseteq wI(f(A), r)$.

Therefore $f(w\alpha I(A, r)) \subseteq wI(f(A), r)$. □

Theorem 4.2.14 Let $f : (X, \mathcal{W}) \to (Y, \mathcal{N})$ be a mapping on two r-FWS's. Then the following statements are hold:

1 If f is fuzzy r-$W^*\alpha$-open, then $f(w\alpha I(A, r)) \subseteq wI(f(A), r)$ for $A \in I^X$.

2 If $f(w\alpha I(A, r)) \subseteq wI(f(A), r)$ for $A \in I^X$ and $\mathcal{N}$ has the property ($\mathcal{U}$), then f is fuzzy r-$W^*\alpha$-open.

Proof. 1. For $A \in I^X$,

$$\begin{aligned} f(w\alpha I(A, r) &= f(\cup\{B \in I^X : B \subseteq A, B \text{ is } \alpha - \mathcal{W}_r \text{ open}\}) \\ &= \cup\{f(B) \in I^Y : f(B) \subseteq f(A), f(B) \text{ is } r \text{ -FWS open}\} \\ &\subseteq \cup\{U \in I^Y : U \subseteq f(A), U \text{ is } r \text{ -FWS open}\} \\ &= wI(f(A), r). \end{aligned}$$

Hence $f(w\alpha I(A, r)) \subseteq wI(f(A), r)$.

2. Let A be α-$\mathcal{W}_r$ open in X. Since $\mathcal{N}$ has the property ($\mathcal{U}$), we have $wI(f(A), r)$ is r-FWS open. From Theorem 4.2.15, we get that f is fuzzy r-$W^*\alpha$-open. □

Theorem 4.2.15 From Theorem 4.2.13, If $f(w\alpha I(A, r)) \subseteq wI(f(A), r)$ and $wI(f(A), r)$ is r-FWS open for all $A \in I^X$, then f is fuzzy r-W^* α-open.

Proof. Let A be r-FWS α-open in X, then $A = w\alpha I(A, r)$.

And $f(A) = f(w\alpha I(A, r)) \subseteq wI(f(A), r) \subseteq f(A)$.

since $wI(f(A), r)$ is r-FWS open, we have $f(A)$ is r-FWS open. □

Theorem 4.2.16 Let $f : (X, \mathcal{W}) \to (Y, \mathcal{N})$ be a mapping on two r-FWS's. and $wC(f(A), r)$ is r-FWS closed. Then the following statements are equivalent:

1 f is fuzzy r-$W^*\alpha$-closed.

2 $wC(f(A), r) \subseteq f(w\alpha C(A, r))$ for $A \in I^X$.

Proof. (1) ⇒ (2) Let $A \in I^X$, then

$f(w\alpha C(A,r)) = f(\cap\{F_\alpha \in I^X : F_\alpha$ is α-$\mathcal{W}_r$ closed and $A \subseteq F_\alpha$, $\alpha \in \Lambda\})$.

$= \cap\{f(F_\alpha) \in I^X : f(F_\alpha)$ is α-$\mathcal{W}_r$ closed and $f(A) \subseteq f(F_\alpha)$, $\alpha \in \Lambda\})$.

$\supseteq \cap\{G_\alpha \in I^X : G_\alpha$ is α-$\mathcal{W}_r$ closed and $f(A) \subseteq G_\alpha$, $\alpha \in \Lambda\})$.

$= wC(f(A), r)$.

(2) ⇒ (1) Let A be α-$\mathcal{W}_r$ open, then $w\alpha C(A, r) = A$.

and $wC(f(A), r) \subseteq f(w\alpha C(A, r)) = f(A) \subseteq wC(f(A), r)$.

Since $wC(f(A), r)$ is closed, $f(A)$ is closed. □

CHAPTER 5

CONCLUSIONS

The aim of this thesis is to introduce the concepts of ordinary smooth r-minimal spaces which study open set, closed set, closure and interior it intersects on such. The studied properties of opens mapping, continuous mapping and compactness. And introduced r-mb generalized closed sets, Also we studied characterization of extremely disconnected and T_{gs} space on ordinary smooth r-minimal spaces. The results are follows:

1 Ordinary Smooth r-Minimal Structure Spaces

Let X be a nonempty set and $r \in (0,1]$. A mapping $\mathcal{M} : 2^X \to I$ is said to have an ordinary smooth r-minimal structure if the family

$$\mathcal{M}_r = \{A \in 2^X : \mathcal{M}(A) \geq r\}.$$

contains $\emptyset$ and X.

Then the $(X, \mathcal{M})$ is called an *ordinary smooth r-minimal structure space* (simply, r-OSMS). Every member of $\mathcal{M}_r$ is called an *ordinary smooth r-minimal open set* (simply, r-OSM open set). A subset A of X is called an *ordinary smooth r-minimal closed set* (simply, r-OSM closed set) if the complement of A (simply, A^C) is an ordinary smooth r-minimal open set.

1.1 Let $(X, \mathcal{M})$ be an r-OSMS and $r \in (0,1]$. The r-OSM closure and the r-OSM interior of A, denoted by $mC(A,r)$ and $mI(A,r)$, respectively, are defined as follows:

1.1.1 $mC(A,r) = \cap\{B \in 2^X : B^C \in \mathcal{M}_r \text{ and } A \subseteq B\}$,

1.1.2 $mI(A,r) = \cup\{B \in 2^X : B \in \mathcal{M}_r \text{ and } B \subseteq A\}$.

1.2 Let $(X, \mathcal{M})$ be an r-OSMS and $A, B \in 2^X$.

1.2.1 $mI(A,r) \subseteq A$.

1.2.2 If A is an r-OSM open set, then $mI(A,r) = A$.

1.2.3 $A \subseteq mC(A,r)$.

1.2.4 If A is an r-OSM closed set, then $mC(A,r) = A$.

1.2.5 If $A \subseteq B$, then $mI(A,r) \subseteq mI(B,r)$ and $mC(A,r) \subseteq mC(B,r)$.

1.2.6 $mI(A,r) \cap mI(B,r) \supseteq mI(A \cap B,r)$ and $mC(A,r) \cup mC(B,r) \subseteq mC(A \cup B,r)$.

1.2.7 $mI(mI(A,r),r) = mI(A,r)$ and $mC(mC(A,r),r) = mC(A,r)$.

1.2.8 $X - mC(A,r) = mI(X - A,r)$ and $X - mI(A,r) = mC(X - A,r)$.

1.3 Let $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ be a mapping on two r-OSMS's. Then f is said to be

1.3.1 r-M continuous mapping if for every $A \in \mathcal{N}_r$, $f^{-1}(A)$ is in $\mathcal{M}_r$.

1.3.2 r-M open mapping if for every $A \in \mathcal{M}_r$, $f(A)$ is in $\mathcal{N}_r$.

1.4 r-OSM compactness.

1.4.1 Let $(X, \mathcal{M})$ be an r-OSMS and $\{A_i \in 2^X : i \in J\}$. A is called an *ordinary smooth r-minimal cover* (simply, r-OSM cover) of X if $\cup\{A_i : i \in J\} = X$. It is an *ordinary smooth r-minimal open cover* (simply, r-OSM open cover) if each A_i is an r-OSM set. $\{B_i \in 2^X : i \in J\}$ is called an *ordinary smooth r-minimal open cover of $B \subseteq X$* if $B \subseteq \cup\{B_i \in 2^X : i \in J\}$.

1.4.2 Let $(X, \mathcal{M})$ be an r-OSMS. An $A \in X$ is said to be *ordinary smooth r-minimal compact* (simply, r-OSM compact) if every r-OSM open cover $\{A_i \in \mathcal{M}_r : i \in J\}$ of A has a finite subcover.

1.4.3 Let $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ be an r-M continuous mapping on two r-OSMS's. If A is an r-OSM compact set, then $f(A)$ is also an r-OSM compact set.

1.4.4 Let $(X, \mathcal{M})$ be an r-OSMS. An $A \in X$ is said to be *ordinary smooth r-minimal almost compact* (simply, r-OSM almost compact) if for every r-OSM open cover $\{A_i \in 2^X :$

$i \in J\}$ of A, there exists $J_0 = \{j_1, j_2, ..., j_n\} \subseteq J$ such that $A \subseteq \bigcup_{i \in J_0} mC(A_i, r)$.

1.4.5 Let $(X, \mathcal{M})$ be an r-OSMS. An $A \in X$ is said to be *ordinary smooth r-minimal nearly compact* (simply, r-OSM nearly compact) if for every r-OSM open cover $\{A_i : i \in J\}$ of A, there exists $J_0 = \{j_1, j_2, ..., j_n\} \subseteq J$ such that $A \subseteq \bigcup_{i \in J_0} mI(mC(A_i, r), r)$.

1.4.6 Let $(X, \mathcal{M})$ be an r-OSMS. If a subset A in X is an r-OSM compact, then it is an r-OSM nearly compact.

1.5 r-M continuous and r-M open.

1.5.1 Let X be a nonempty set and $\mathcal{M} : 2^X \to I$ a family on X. The family $\mathcal{M}$ is said to have the property $(\mathcal{U})$ if for $A_i \in \mathcal{M}_r (i \in J)$

$$\mathcal{M}(\cup A_i) \geq \wedge \mathcal{M}(A_i).$$

1.5.2 Let $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ be a mapping on two r-OSMS's. Then the following statements are equivalent:

(i) f is r-M continuous.

(ii) $f^{-1}(B)$ is an r-OSM closed set, for each r-OSM closed set B in Y.

1.5.3 Let $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ be a mapping on two r-OSMS's. Then the following statements are hold:

(i) If f is r-M continuous, then $f(mC(A, r)) \subseteq mC(f(A), r)$ for all $A \in 2^X$.

(ii) If $f^{-1}(mI(B, r)) \subseteq mI(f^{-1}(B), r)$, for all $B \in 2^Y$ is true and $\mathcal{M}$ has the property $(\mathcal{U})$, then f is r-M continuous.

1.5.4 Let $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ be a mapping on two r-OSMS's. Then the following statements are equivalent:

(i) $f(mC(A, r)) \subseteq mC(f(A), r)$ for $A \in 2^X$.

(ii) $mC(f^{-1}(B), r) \subseteq f^{-1}(mC(B, r))$ for $B \in 2^Y$.

(iii) $f^{-1}(mI(B,r)) \subseteq mI(f^{-1}(B),r)$ for $B \in 2^Y$.

1.5.5 Let $f : (X,\mathcal{M}) \to (Y,\mathcal{N})$ be a mapping on two r-OSMS's. Then

(i) $f(mI(A,r)) \subseteq mI(f(A),r)$ for $A \in 2^X$.

(ii) $mI(f^{-1}(B),r) \subseteq f^{-1}(mI(B,r))$ for $B \in 2^Y$.

1.5.6 Let $f : (X,\mathcal{M}) \to (Y,\mathcal{N})$ be a mapping on two r-OSMS's. Then the following statements are equivalent:

(i) If f is r-M open, then $f(mI(A,r)) \subseteq mI(f(A),r)$ for $A \in 2^X$.

(ii) If $mI(f^{-1}(B),r) \subseteq f^{-1}(mI(B,r))$ for $B \in 2^Y$ and $\mathcal{N}$ has property ($\mathcal{U}$), then f is r-M open.

1.5.7 Let $f : (X,\mathcal{M}) \to (Y,\mathcal{N})$ be a mapping on two r-OSMS's. Then f is said to be *ordinary smooth weakly r-M continuous* (simply,r-M weak continuous) if for $x \in X$ and each r-OSM open set V containing $f(x)$, there is an r-OSM open set U containing x such that $f(U) \subseteq mC(V,r)$.

1.5.8 Let $f : (X,\mathcal{M}) \to (Y,\mathcal{N})$ be a mapping on two r-OSMS's. Then the following statements are equivalent:

(i) $f^{-1}(V) \subseteq mI(f^{-1}(mC(V,r)),r)$ for each r-OSM open set V in Y.

(ii) $mC(f^{-1}(mI(B,r),r) \subseteq f^{-1}(B)$ for each r-OSM closed set B in Y.

(iii) $mC(f^{-1}(B),r) \subseteq f^{-1}(mC(B,r))$ for each r-OSM open set B in Y.

1.5.9 Let $f : (X,\mathcal{M}) \to (Y,\mathcal{N})$ be a mapping on two r-OSMS's. Then the following statements are hold:

(i) If f is r-M weak continuous, then $f^{-1}(V) \subseteq mI(f^{-1}(mC(V,r)),r)$ for each r-OSM open set V in Y.

(ii) If $mC(f^{-1}(B),r) \subseteq f^{-1}(mC(B,r))$ for each r-OSM open set B in Y is true and $\mathcal{N}$ has the property ($\mathcal{U}$), then

f is r-M weak continuous.

1.5.10 Let $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ be a mapping on two r-OSMS's and $A \in 2^Y$. If f is r-M weak continuous, then the following statements hold :

(i) $f^{-1}(A) \subseteq mI(f^{-1}(mC(A, r)), r)$ for $A = mI(A, r)$.

(ii) $mC(f^{-1}(mI(A, r)), r) \subseteq f^{-1}(A)$ for $A = mC(A, r)$.

1.5.11 Let $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ be a mapping on two r-OSMS's. Then f is said to be *ordinary smooth almost* $r-M$ *continuous* (simply, r-M almost continuous) if for $x \in X$ and each r-OSM open set V containing $f(x)$, there is an r-OSM open set U containing x such that $f(U) \subseteq mI(mC(V, r), r)$.

1.5.12 Let $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ be a mapping on two r-OSMS's. Then the following statements are equivalent:

(i) f is $r - M$ almost continuous.

(ii) $f^{-1}(B) \subseteq mI(f^{-1}(mI(mC(B, r), r)), r)$ for each r-OSM open set B in Y.

(iii) $mC(f^{-1}(mC(mI(F, r), r)), r) \subseteq f^{-1}(F)$ for each r-OSM closed set F in Y.

1.6 Relationships between some types of r-OSM compactness and r-M continuous.

1.6.1 Let $(X, \mathcal{M})$ be an r-OSMS. If a subset A in X is r-OSM compact, then it is also r-OSM almost compact.

1.6.2 Let $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ be an r-M continuous mapping on two r-OSMS's. If A is an r-OSM almost compact set, then $f(A)$ is also an r-OSM almost compact set.

1.6.3 Let $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ be an r-M continuous and r-M open mapping on two r-OSMS's. If A is an r-OSM nearly compact set, then $f(A)$ is an r-OSM nearly compact set.

1.6.4 Let $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ be an r-M weak continuous mappings on two r-OSMS's. If A is an r-OSM compact set in

X and $\mathcal{M}$ has property $(\mathcal{U})$, then $f(A)$ is an r-OSM almost compact set.

1.6.5 Let $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ be an r-M weak continuous mappings on two r-OSMS's. If A is an r-OSM almost compact set and $\mathcal{M}$ has property $(\mathcal{U})$, then $f(A)$ is an r-OSM almost compact set.

1.6.6 Let $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ be an r-M weak continuous mappings on two r-OSMS's. If A is an r-OSM nearly r-minimal compact set and $\mathcal{M}$ has property $(\mathcal{U})$, then $f(A)$ is an r-OSM nearly compact set.

1.6.7 Let $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ be an r-M almost continuous mappings on two r-OSMS's. If A is an r-OSM compact set in X and $\mathcal{M}$ has property $(\mathcal{U})$, then $f(A)$ is an r-OSM nearly compact set.

1.6.8 Let $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ be an r-M almost continuous and r-M open mappings on two r-OSMS's. If A is an r-OSM almost compact set and $\mathcal{M}$ has property $(\mathcal{U})$, then $f(A)$ is an r-OSM almost compact set.

1.6.9 Let $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ be an r-M almost continuous and r-M open bmappings on two r-OSMS's. If A is an r-OSM compact set and $\mathcal{M}$ has property $(\mathcal{U})$, then $f(A)$ is an r-OSM compact set.

1.7 Let X be a nonempty set and $\mathcal{M} : 2^X \to I$ a family on X. The family $\mathcal{M}$ is said to have the property $(\mathcal{U})$ if for $A_i \in \mathcal{M}_r (i \in J)$

$$\mathcal{M}(\cup A_i) \geq \wedge \mathcal{M}(A_i).$$

1.8 Let $(X, \mathcal{M})$ be an r-OSMS with the property $(\mathcal{U})$. Then

1.8.1 $mI(A, r) = A$ if and only if $A \in \mathcal{M}_r$ for $A \in 2^X$.

1.8.2 $mC(A, r) = A$ if and only if $A^C \in \mathcal{M}_r$ for $A \in 2^X$.

1.9 On Generalized r-mb Closed sets.

1.9.1 Let $(X, \mathcal{M})$ be an r-OSMS and $A \in 2^X$. Then A is called:

(i) ordinary smooth r-minimal semi-closed (briefly r-ms closed)

if $mI(mC(A,r),r) \subseteq A$

(ii) ordinary smooth r-minimal pre-closed (briefly r-$mpre$ closed)

if $mC(mI(A,r),r) \subseteq A$

(iii) ordinary smooth r-minimal b-closed (briefly r-mb closed)

if $(mC(mI(A,r),r) \cap mI(mC(A,r),r)) \subseteq A$

(iv) ordinary smooth r-minimal semi-preclosed (briefly r-msp closed)

if $mI(mC(mI(A,r),r),r) \subseteq A$

1.9.2 Let $(X, \mathcal{M})$ be an r-OSMS and $A \in 2^X$. Then A is called:

(i) r-mgb closed if $bmC(A,r) \subseteq U$ whenever $A \subseteq U$ and $U \in \mathcal{M}_r$.

(ii) r-msg closed if $smC(A,r) \subseteq U$ whenever $A \subseteq U$ and $U \in r$-$mSO(X)$.

(iii) r-mgs closed if $smC(A,r) \subseteq U$ whenever $A \subseteq U$ and $U \in \mathcal{M}_r$.

(iv) r-mgp closed if $pmC(A,r) \subseteq U$ whenever $A \subseteq U$ and $U \in \mathcal{M}_r$.

(v) r-$mgsp$ closed if $spmC(A,r) \subseteq U$ whenever $A \subseteq U$ and $U \in \mathcal{M}_r$.

1.9.3 Let $(X, \mathcal{M})$ be an r-OSMS and $A \in 2^X$. Then A is called **nowhere dense** if and only if $mI(mC(A,r),r) = \emptyset$.

1.9.4 Let $(X, \mathcal{M})$ be an r-OSMS and $D \in 2^X$. Then D is called **dense** if and only if $mC(D,r) = X$.

1.9.5 Let $(X, \mathcal{M})$ be an r-OSMS and $E \in 2^X$. Then E is called **codense** if and only if $mI(E,r) = \emptyset$.

1.9.6 Let $(X, \mathcal{M})$ be an r-OSMS is said to be:

(i) T_{gs} if every gs-closed subset of X is sg-closed.

(ii) Extremely disconnected if the closure of each open subsets of X is open.

1.9.7 Let $(X, \mathcal{M})$ be an r-OSMS and $A \in 2^X$. Then A is called :

(i) ordinary smooth r-minimal semi-open (briefly r-ms open) if $A \subseteq mC(mI(A, r), r)$

(ii) ordinary smooth r-minimal regular open (briefly r-mrg open)
if $A = mI(mC(A, r), r)$

(iii) ordinary smooth r-minimal pre-open (briefly r-$mpre$ open) if $A \subseteq mI(mC(A, r), r)$

1.9.8 Let $(X, \mathcal{M})$ be an r-OSMS and $A \in 2^X$. Then A is called :

(i) $pmC(A, r) = A \cup mC(mI(A, r), r)$

(ii) $smC(A, r) = A \cup mI(mC(A, r), r)$

1.9.9 Let $(X, \mathcal{M})$ be an r-OSMS and $r \in (0, 1]$.

(i) $smC(A, r) = \cap\{B \in 2^X : B$ is s-closed set and $A \subseteq B\}$

(ii) $pmC(A, r) = \cap\{B \in 2^X : B$ is pre-closed set and $A \subseteq B\}$

(iii) $bmC(A, r) = \cap\{B \in 2^X : B$ is b-closed set and $A \subseteq B\}$

(iv) $spmC(A, r) = \cap\{B \in 2^X : B$ is sp-closed set and $A \subseteq B\}$

1.10 r-mgb Closed Sets and Their Relationships.

1.10.1 If A_α is a r-mb closed set for $\alpha \in \Lambda$, then $\bigcap_{\alpha \in \Lambda} A_\alpha$ is a r-mb closed set

1.10.2 If A_α is r-msp closed for all $\alpha \in \Lambda$, then $\bigcap_{\alpha \in \Lambda} A_\alpha$ is r-msp closed.

1.10.3 If A_α is r-ms closed for all $\alpha \in \Lambda$, then $\bigcap_{\alpha \in \Lambda} A_\alpha$ is r-ms closed.

1.10.4 If A_α is r-$mpre$ closed for all $\alpha \in \Lambda$, then $\bigcap_{\alpha \in \Lambda} A_\alpha$ is r-$mpre$ closed.

1.10.5 Every r-mb closed set is r-mgb closed.

1.10.6 Every r-msp closed set is r-$mgsp$ closed.

1.10.7 Every r-msg closed set is r-mgs closed.

1.10.8 Every r-$mpre$ closed set is r-mgp closed.

1.10.9 Every r-ms closed set is r-msg closed.

1.10.10 Every r-mgp closed set is r-$mgsp$ closed.

1.10.11 Every r-ms closed set is r-mgs closed.

1.10.12 Every r-mgp closed set is r-mgb closed.

1.10.13 Every r-mgb closed set is r-$mgsp$ closed.

1.10.14 If G is r-mrg open set , then G is r-OSM open.

1.10.15 If G is r-mrg open set, then G is r-OSM closed.

1.10.16 If F is r-ms closed set, then F is r-msp closed.

1.10.17 If G is r-OSM open set , then G is r-ms open.

1.10.18 If F is r-ms closed set, then F is r-mb closed.

1.10.19 If F is r-$mpre$ closed set, then F is r-mb closed.

1.10.20 If F is r-$mpre$ closed set, then F is r-msp closed.

1.10.21 If F is r-mb closed set, then F is r-msp closed.

1.10.22 If X is nowhere dense, then $X - \{x\}$ is r-ms open sets.

1.11 Let X be any r-OSMS, then the following are equivalent:

1.11.1 Every r-mgb closed set is r-mgp closed.

1.11.2 Every r-mb closed set is r-mgp closed.

1.12 Let $(X, \mathcal{M})$ be an r-OSMS the following are equivalent:

1.12.1 Every r-$mgsp$ closed set is r-mgp closed.

1.12.2 Every r-msp closed set is r-mgp closed.

1.12.3 Every r-$mgsp$ closed set is r-mgp closed.

1.12.4 Every r-msg closed set is r-mgp closed.

1.12.5 Every r-msp closed is r-$mpre$ closed.

1.12.6 Every r-mgp closed set is r-mgp closed.

1.12.7 Every r-mb closed set is r-mgp closed.

1.12.8 X is extremely disconnnected.

1.13 Let $(X, \mathcal{M})$ be r-OSMS and let $X_1, X_2 \subseteq X$ defined by $X_1 = \{x \in X : \{x\}$ is nowhere dense$\}$ and $X_2 = \{x \in X : \{x\}$ is r-$mpre$ open$\}$. It is easy to see that $\{X_1, X_2\}$ is a decomposition of X (i.e. $X = X_1 \cup X_2$).

1.14 Let $(X, \mathcal{M})$ be r-OSMS and $A \in 2^X$. Then A is r-msg closed if and only if $X_1 \cap smC(A, r) \subseteq A$

1.15 Let X be any r-OSMS, X is T_{gs} if and only if every singleton is either r-$mpre$ open or r-OSM closed.

1.16 Let $(X, \mathcal{M})$ be an r-OSMS, every singleton is either r-$mpre$ open or nowhere dense.

1.17 Let $(X, \mathcal{M})$ be an r-OSMS , the following are equivalent.

1.17.1 Every r-mgb closed set is r-mb closed.

1.17.2 Every r-mgs closed set is r-mb closed.

1.17.3 Every r-mgb closed set is r-msp closed.

1.17.4 Every r-mgp closed set is r-$mpre$ closed.

1.17.5 Every r-$mgsp$ closed set is r-msp closed.

1.17.6 Every r-mgp closed set is r-msp closed.

1.17.7 X is T_{gs}

1.18 Let $(X, \mathcal{M})$ be an r-OSMS , the following are equivalent.

1.18.1 Evaery r-mb closed set is r-mgs closed.

1.18.2 Every r-mgb closed set is r-mgs closed.

1.19 If every r-mgp closed set of a space X is r-msg closed, then X is T_{gs}.

1.19.1 Every r-$mgsp$ closed set is r-$mpre$ closed.

1.19.2 Every r-mgb closed set is r-$mpre$ closed.

1.19.3 X is extremely disconnected and T_{gs}.

1.20 Let $(X, \mathcal{M})$ be r-OSMS. Then every r-msg closed set is r-mb closed.

2 Fuzzy r-Weak Structure Spaces

Let X be a nonempty set and $r \in (0,1]$. A fuzzy family $\mathcal{W} : I^X \to I$ on X is said to have a fuzzy r-weakly structure if the family

$$\mathcal{W}_r = \{A \in I^X : \mathcal{W}(A) \geq r\}$$

contains $\widetilde{0}$.

Then the pair $(X, \mathcal{W})$ is called a *fuzzy r-weakly structure space* (simply, r-FWS). Every member of $\mathcal{W}_r$ is called a *fuzzy r-weak open set* (simply, r-FWS open set). A fuzzy set A is called a *fuzzy r-weak closed set* (simply, r-FWS closed set) if the complement of A (simply, A^C) is a fuzzy r-weak open set.

2.1 Let $(X, \mathcal{W})$ be an r-FWS and $r \in (0,1]$.

2.1.1 $wC(A, r) = \cap\{B \in I^X : B^C \in \mathcal{W}_r$ and $A \subseteq B\}$

2.1.2 $wI(A, r) = \cup\{B \in I^X : B \in \mathcal{W}_r$ and $B \subseteq A\}$

2.2 Let $(X, \mathcal{W})$ be an r-FWS and $A, B \in I^X$.

2.2.1 $wI(A, r) \subseteq A$ and if $A \in \mathcal{W}_r$, then $wI(A, r) = A$.

2.2.2 $A \subseteq wC(A, r)$ and if $A^C \in \mathcal{W}_r$, then $wC(A, r) = A$.

2.2.3 If $A \subseteq B$, then $wI(A, r) \subseteq wI(B, r)$ and $wC(A, r) \subseteq wC(B, r)$.

2.2.4 $wI(A, r) \cap wI(B, r) \supseteq wI(A \cap B, r)$ and $wC(A, r) \cup wC(B, r) \subseteq wC(A \cup B, r)$.

2.2.5 $wI(mI(A, r), r) = wI(A, r)$ and $wC(wC(A, r), r) = wC(A, r)$.

2.2.6 $\widetilde{1} - wC(A, r) = wI(\widetilde{1} - A, r)$ and $\widetilde{1} - wI(A, r) = wC(\widetilde{1} - A, r)$.

2.3 Let $(X, \mathcal{W})$ be an r-FWS and $A \in I^X$. Then a fuzzy set A is called a *fuzzy r-weak semiopen set* (simply, r-FWS semiopen set) in X if

$$A \subseteq wC(wI(A, r), r).$$

A fuzzy set A is called a *fuzzy r-weak semiclosed set* (simply, r-FWS semiclosed set) if the complement of A is fuzzy r-weak semiopen.

Let $(X, \mathcal{W})$ and $(Y, \mathcal{N})$ be two r-FWS's. Then $f : X \to Y$ is said to be *fuzzy r-W continuous function* if for every $A \in \mathcal{N}_r$, $f^{-1}(A)$ is in $\mathcal{W}_r$.

2.4 Let $(X, \mathcal{W})$ be an r-FWS and $A \in I^X$. Then a fuzzy set A is called a *fuzzy r-weak α-open set* (simply, α-$\mathcal{W}_r$ open set) in X if

$$A \subseteq wI(wC(wI(A, r), r), r).$$

A fuzzy set A is called a *fuzzy r-weak α-closed set* (simply, α-$\mathcal{W}_r$ closed set) if the complement of A is fuzzy r-weak α-open.

2.5 Let $(X, \mathcal{W})$ be an r-FWS. Then the following condition are hold:

2.5.1 Every r-FWS open set is α-$\mathcal{W}_r$ open.

2.5.2 Every α-$\mathcal{W}_r$ open set is r-FWS semiopen.

2.6 The following implications are obtained but the converses are not trure in general.

fuzzy r-weak open $\Rightarrow$ fuzzy r-weak α-open $\Rightarrow$ fuzzy r-weak semiopen.

3 Let $(X, \mathcal{W})$ be an r-FWS. Then a fuzzy set A is α-$\mathcal{W}_r$ closed set if and only if $wC(wI(wC(A, r), r), r) \subseteq A$.

4 Let $(X, \mathcal{W})$ be an r-FWS. Then any union of α-$\mathcal{W}_r$ open set isα-$\mathcal{W}_r$ open.

5 Let $(X, \mathcal{W})$ be an r-FWS. For any $A \in I^X, w\alpha C(A, r)$ and $w\alpha I(A, r)$, respectively, are defined as the following

$$w\alpha C(A, r) = \cap\{F \in I^X : A \subseteq F, F \text{ is } r\text{-FWS } \alpha\text{-closed }\};$$

$$w\alpha I(A, r) = \cup\{U \in I^X : U \subseteq A, U \text{ is } r\text{-FWS } \alpha\text{-open }\}.$$

6 Let $(X, \mathcal{W})$ be an r-FWS and $A \in I^X$. Then the following statments are hold.

6.1 $w\alpha I(A, r) \subseteq A$.

6.2 If $A \subseteq B$, then $w\alpha I(A, r) \subseteq w\alpha I(B, r)$.

6.3 A is α-$\mathcal{W}_r$ open if and only if $w\alpha I(A, r) = (A, r)$.

6.4 $w\alpha I(w\alpha I(A, r), r) = w\alpha I(A, r)$.

6.5 $w\alpha C(\widetilde{1} - A, r) = \widetilde{1} - w\alpha I(A, r)$ and $w\alpha I(\widetilde{1} - A, r) = \widetilde{1} - w\alpha C(A, r)$.

7 Let $(X, \mathcal{W})$ be an r-FWS and $A \in I^X$. Then

7.1 $A \subseteq w\alpha C(A, r)$.

7.2 If $A \subseteq B, then w\alpha C(A, r) \subseteq w\alpha C(B, r)$

7.3 A is α-$\mathcal{W}_r$ closed if and only if $w\alpha C(A, r) = (A, r)$.

7.4 $w\alpha C(w\alpha I(A, r), r) = w\alpha C(A, r)$.

8 Let $f : (X, \mathcal{W}) \to (Y, \mathcal{N})$ be a mapping on two r-FWS's. Then f is said to be *fuzzy r-W continuous function* if for every $A \in \mathcal{N}_r$, $f^{-1}(A)$ is in $\mathcal{W}_r$.

9 Let $f : (X, \mathcal{W}) \to (Y, \mathcal{N})$ be a mapping on two r-FWS's. Then f is said to be *fuzzy r-W α-continuous* if $f^{-1}(U)$ is an α-$\mathcal{W}_r$ open set for all r-FWS open set U in Y.

10 Let $f : (X, \mathcal{W}) \to (Y, \mathcal{N})$ be a mapping on two r-FWS's. Then f said to be *fuzzy r-W semicontinuous* if $f^{-1}(U)$ is an r-FWS semiopen set for all r-FWS open set U in Y.

11 Let $(X, \mathcal{W})$ be an r-FWS then the following condition is true.

11.1 Every fuzzy r-W continuous is fuzzy r-W α-continuous.

11.2 Every fuzzy r-W α-continuous is fuzzy r-W semicontinuous.

12 It is obvious that every fuzzy r-W α-continuous mapping is fuzzy r-W semicontinuous but the converse may not be true as show in the next example. fuzzy r-W continuous $\Rightarrow$ fuzzy r-W α-continuous $\Rightarrow$ fuzzy r-W semicontinuous.

13 Let $f : (X, \mathcal{W}) \to (Y, \mathcal{N})$ be a mapping on two r-FWS's. Then the following statements are equivalent:

13.1 f is fuzzy r-W α-continuous.

13.2 $f^{-1}(B)$ is an α-$\mathcal{W}_r$ closed set for each r-FWS closed set B in Y.

13.3 $f(w\alpha C(A, r)) \subseteq wC(f(A), r)$ for $A \in I^X$.

13.4 $w\alpha C(f^{-1}(B), r) \subseteq f^{-1}(wC(B, r))$ for $B \in I^Y$.

13.5 $f^{-1}(wI(B,r)) \subseteq w\alpha I(f^{-1}(B),r)$ for $B \in I^Y$.

Fuzzy r-W α-open.

14 Let $f:(X,\mathcal{W}) \to (Y,\mathcal{N})$ be a mapping on two r-FWS's. Then

14.1 f is said to be *fuzzy r-W α-open* if for r-FWS open set A in X, $f(A)$ is r-FWS α-open in Y;

14.2 f is said to be *fuzzy r-W α-closed* if for r-FWS closed set A in X, $f(A)$ is r-FWS α-closed in Y.

15 Let $f:(X,\mathcal{W}) \to (Y,\mathcal{N})$ be a mapping on two r-FWS's. Then the following are equivalent:

15.1 f is fuzzy r-W α-open.

15.2 $f(wI(A,r)) \subseteq w\alpha I(f(A),r)$ for $A \in I^X$.

15.3 $wI(f^{-1}(B),r) \subseteq f^{-1}(w\alpha I(B,r))$ for $B \in I^Y$.

16 Let $f:(X,\mathcal{W}) \to (Y,\mathcal{N})$ be a mapping on two r-FWS's. Then the following are equivalent:

16.1 f is fuzzy r-M α-closed.

16.2 $w\alpha C(f(A),r)) \subseteq f(wC(A,r))$ for $A \in I^X$.

17 Let $f:(X,\mathcal{W}) \to (Y,\mathcal{N})$ be a mapping on two r-FWS's. Then

17.1 f is said to be *fuzzy r-$W^*\alpha$-open* if for r-FWS open set A in X, $f(A)$ is r-FWS open in Y;

17.2 f is said to be *fuzzy r-$W^*\alpha$-closed* if for r-FWS closed set A in X, $f(A)$ is r-FWS closed in Y.

18 Let $f:(X,\mathcal{W}) \to (Y,\mathcal{N})$ be a mapping on two r-FWS's.

18.1 f is fuzzy r-$W^*\alpha$-open.

18.2 $fw\alpha I(A,r)) \subseteq wI(f(A),r)$ for $A \in I^X$.

18.3 $w\alpha I(f^{-1}(B),r) \subseteq f^{-1}(wI(B,r))$ for $B \in I^Y$.

Then (1) ⇒ (2) ⇒ (3).

19 Let $f : (X, \mathcal{W}) \to (Y, \mathcal{N})$ be a mapping on two r-FWS's. Then the following are equivalent:

19.1 f is fuzzy $r - M^*\alpha$-closed.

19.2 $wC(f(A), r)) \subseteq f(w\alpha C(A, r)$ for $A \in I^X$.

20 Let X be a nonempty set and $\mathcal{W} : I^X \to I$ a fuzzy family on X. The fuzzy r-weak structure $\mathcal{W}_r$ is said to have the property $(\mathcal{U})$ if for $A_i \in \mathcal{W}_r(i \in J)$,

$$\mathcal{W}(\cup A_i) \geq \wedge \mathcal{W}(A_i).$$

21 Let $(X, \mathcal{W})$ be an r-FMS with the property $(\mathcal{U})$. Then for $A \in I^X$, $wI(A, r) = A$ if and only if A is r-FWS open.

22 Let $f : (X, \mathcal{W}) \to (Y, \mathcal{N})$ be a mapping on two r-FWS's.
If (Y, N) has the property $(\mathcal{U})$, then the following are equivalent:

22.1 f is fuzzy r-$W^*\alpha$-open.

22.2 $fw\alpha I(A, r)) \subseteq wI(f(A), r)$ for $A \in I^X$.

22.3 $w\alpha I(f^{-1}(B), r) \subseteq f^{-1}(wI(B, r))$ for $B \in I^Y$.

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