SEPARATION AXIOMS IN ORDINARY SMOOTH TOPOLOGICAL SPACES

BY

SAKCHAI PHALEE

A thesis submitted in partial fulfillment of the requirements for the degree of Master of Science in Mathematics at Mahasarakham University

July 2016

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The examining committee has unanimously approved this thesis, submitted by Mr. Sakchai Phalee, as a partial fulfillment of the requirements for the Master of Science in Mathematics at Mahasarakham University.

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Sakchai Phalee

ชื่อเรื่อง สัจพจน์การแยกในปริภูมิเชิงทอพอโลยีแบบเรียบสามัญ
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บทคัดย่อ

การศึกษาสัจพจน์การแยกในปริภูมิเชิงทอพอโลยี ได้รับแนวคิดจากสัจพจน์การแยกบนเซตของจำนวนจริง สัจพจน์การแยกสามารถจำแนกปริภูมิเชิงทอพอโลยีชนิดต่าง ๆ ที่สอดคล้องกับเงื่อนไขของการแยกได้เป็นปริภูมิ T_0, T_1, T_2, T_3, T_4 และ T_5 เป็นต้น

ในงานวิจัยนี้ผู้วิจัยได้กำหนดสัจพจน์การแยกบนปริภูมิเชิงทอพอโลยีแบบเรียบสามัญ ซึ่งสามารถจำแนกชนิดของปริภูมิเชิงทอพอ โลยีแบบเรียบสามัญที่สอดคล้องกับเงื่อนไขของการแยกได้เป็นปริภูมิ $OT_0, OT_1, OT_2, OT_3, OT_4$ และ OT_5 และยังเป็นวางนัยทั่วไปของปริภูมิเชิง ทอพอโลยี พร้อมทั้งศึกษาสมบัติของสัจพจน์การแยกและศึกษาฟังก์ชันบนปริภูมิดังกล่าว นอกจากนี้ ผู้วิจัยยังได้นำเสนอแนวคิดเกี่ยวกับเซตหนาแน่น OST บนปริภูมิเชิงทอพอโลยีแบบเรียบสามัญ และศึกษาสมบัติของเซตหนาแน่น OST

คำสำคัญ : ปริภูมิเชิงทอพอโลยีแบบเรียบสามัญ; ส่วนปิดคลุมแบบเรียบสามัญ; ภายในแบบเรียบสามัญ; ฟังก์ชันต่อเนื่องแบบเรียบสามัญ; ฟังก์ชันเปิดแบบเรียบสามัญ; ฟังก์ชันปิดแบบเรียบสามัญ

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ABSTRACT

The concepts of separation axioms in classical topological spaces were established from separation axioms such concept were characterization for instance T_0, T_1, T_2, T_3, T_4 and T_5 spaces.

In this research, we determine the separation axioms and identify other type of ordinary smooth topological spaces which characterization type of ordinary smooth topological spaces which we call $OT_0, OT_1, OT_2, OT_3, OT_4$ and OT_5 spaces. These spaces can be generalized the separation axioms of classical topological spaces. We also study some properties of separation axioms and study functions on these spaces. Moreover, we introduce the concepts of OST-dense sets in ordinary smooth topological spaces and study the basic properties of OST-dense sets.

Keywords : Ordinary smooth topological spaces; Ordinary smooth closure; Ordinary smooth interior; Ordinary smooth continuous; Ordinary smooth open; Ordinary smooth closed.

CONTENTS

CHAPTER 1

INTRODUCTION

The concepts of fuzzy topology on a set X which consists of a set X and structure T on X and this spaces called the fuzzy topological spaces (briefly *fts*) was first introduced by Chang [1]. In 1986, Badard [2] introduced the concepts of smooth topological spaces and redefined fuzzy topology was called smooth topology (briefly *st*) and this space called the smooth topological spaces (briefly *sts*). In 1992, Ramadan [3] rediscover the smooth topological spaces.

In 2001, El-gayyar, Kerre and Ramadan [4] introduced concepts of separation axioms in smooth topological spaces and investigated some of their properties and the relations between them in smooth topological spaces. Next, the concepts of ordinary smooth topology (briefly OST) on a set X and notion of ordinary smooth continuity were introduced by Lim, Ryoo and Hur in [5]. They also studied and investigated some properties of ordinary smooth subspaces. After that, they introduced the notions of ordinary smooth closure and ordinary smooth interior of ordinary subsets and investigated some of their properties, also they introduced ordinary smooth open preserving functions and studied some of their properties. In addition, they developed the notions of ordinary smooth compactness, ordinary smooth almost compactness, and ordinary near compactness and discusses them in the general framework on ordinary smooth topological spaces in [7].

In 2013, Lee, Lim and Hur [6] redefined the notions of ordinary smooth closure and ordinary smooth interior. Also they introduced and studied some of their properties of compact in an ordinary smooth topological spaces, and redefined a new definition of ordinary smooth closure and ordinary smooth interior.

For our purpose, we introduce the concepts of some separation axioms in ordinary smooth topological spaces and study some properties of these spaces. Moreover, we study some properties of functions on ordinary smooth topological spaces. Furthermore, we introduce the concepts of OST-dense sets in ordinary smooth topological spaces and study the basic properties of OST-dense sets in ordinary smooth topological spaces.

In the first chapter, the introduction was present.

In Chapter 2, we present some basis concepts and results of ordinary smooth topology without proofs which are needed in the subsequent chapters.

In Chapter 3, we introduce the concepts of some separation axioms in ordinary smooth topological spaces and study some properties on the spaces. Moreover, we study some properties of functions on ordinary smooth topological spaces.

In Chapter 4, we introduce the concepts of OST-dense sets in ordinary smooth topological spaces and study the basic properties of OST-dense sets.

In the last Chapter, we summarize results of our study.

CHAPTER 2

PRELIMINARIES

In this chapter, we will recall some definitions, notations, dealing with some preliminaries and some useful results that will be duplicated in later chapter.

2.1 Classical topological spaces

In this section we discuss some properties of classical topological spaces and some properties of closure, interior, continuous functions and separation axioms of all those. First of all, we will recall the definition of classical topological spaces.

Definition 2.1.1. [8] Let X be a nonempty set. A class of τ of subsets of X is a *classical topology* on X if and only if τ satisfies the following axioms:

(i) $X, \emptyset \in \tau$.

(ii) If $A_1, A_2 \in \tau$, then $A_1 \cap A_2 \in \tau$.

(iii) If $A_\alpha \in \tau$ for all $\alpha \in \Gamma$, then $\bigcup\limits_{\alpha\in\Gamma} A_\alpha \in \tau$.

The pair (X, τ) is called a *classical topological space* and the members of τ are called *open sets*.

The operators on X which induced by the topologies τ are follows:

Definition 2.1.2. [8] Let (X, τ) be a classical topological spaces and let $A \subseteq X$. Then *closure* of A in X, denoted by $\overline{A}$, is defined by

$$\overline{A} = \bigcap\{F : A \subseteq F, X\backslash F \in \tau\}.$$

Definition 2.1.3. [8] Let (X, τ) be a classical topological spaces and let $A \subseteq X$. Then *interior* of A in X , denoted by A° , is defined by

$$A^\circ = \bigcup\{U : U \subseteq A, U \in \tau\}.$$

The function $f : X \to Y$ which pre-image preserves open set, preserves open set and preserves closed set are called continuous, open function and closed function respectively.

Definition 2.1.4. [8] Let (X,τ_1) and (Y,τ_2) be two classical topological spaces. Then a mapping $f:X\to Y$ is said to be *continuous* if $U\in\tau_2$ implies that $f^{-1}(U)\in\tau_1$.

Definition 2.1.5. [8] Let (X,τ_1) and (Y,τ_2) be two classical topological spaces. Then a mapping $f:X\to Y$ is said to be

(i) an *open function* if and only if $f(G)$ is an open subsets in Y for all open subset G in X.

(ii) a *closed function* if and only if $f(F)$ is a closed subsets in Y for all closed subset F in X.

Definition 2.1.6. [8] Let (X,τ_1) and (Y,τ_2) be two classical topological space. Then a mapping $f:X\to Y$ is called a homeomorphism if and only if

(i) f is a bijective,

(ii) f and f^{-1} are continuous.

Next, we will recall the definitions of $T_0,T_1,...,T_4$ and T_5-spaces and dense sets in classical topological spaces.

Definition 2.1.7. [8] A topological space X is a T_0- *space* if and only if for any pair of distinct points $a,b\in X$, there exists an open set U such that either $a\in U$ and $b\notin U$ or $b\in U$ and $a\notin U$ (i.e. U containing exactly one of these points).

Definition 2.1.8. [8] A topological space X is a T_1-*space* if and only if for any pair of distinct points $a,b\in X$, there exist an open sets U and V such that $a\in U,\ b\notin U$ and $b\in V,\ a\notin V$.

Definition 2.1.9. [8] A topological space X is a T_2-*space* if and only if for any pair of distinct points $a,b\in X$, there exist an open sets U and V such that $a\in U,\ b\in V$ and $U\cap V=\emptyset$.

Definition 2.1.10. [8] A topological space X is a T_3-*space* if and only if for any closed subset A of X and b is a point in X with $b\notin A$, there exist disjoint open sets U and V such that $A\subseteq U$ and $b\in V$.

Definition 2.1.11. [8] A topological space X is a T_4-*space* if and only if for any pair of distinct closed subsets A and B in X, there exist disjoint open set U and V such that $A\subseteq U,\ B\subseteq V$ and $U\cap V=\emptyset$.

Definition 2.1.12. [8] A topological space X is a T_5-*space* if and only if for separated set A and B in X(i.e., $A \cap \overline{B} = \emptyset$ and $\overline{A} \cap B = \emptyset$), there exist disjoint open set U and V such that $A \subseteq U,\ B \subseteq V$ and $U \cap V = \emptyset$.

Definition 2.1.13. [8] Let (X, τ) be a classical topological space and A be a subset of X. A is called a *dense set* in X if $X = \overline{A}$.

2.2 Smooth topological spaces

In this section we discuss some properties of smooth topological spaces and some properties of smooth closure, smooth interior, smooth continuous and some separation axioms in smooth topological spaces.

Definition 2.2.1. [1] For a set X, we define a *fuzzy set* in X to be function $\mu : X \to [0, 1]$.

For each a nonempty set X, let I^X be the family of all fuzzy sets on X and I be the closed interval [0, 1]. And intersections and union of fuzzy sets are denoted by $\wedge$ and $\vee$, respectively, and defined by

$$\wedge A_i = \text{inf}\{A_i(x) : i \in J \text{ and } x \in X\}.$$

$$\vee A_i = \text{sup}\{A_i(x) : i \in J \text{ and } x \in X\}.$$

First of all, we will recall smooth topological spaces.

Definition 2.2.2. [3] Let X be a nonempty set. Then a mapping $\tau' : I^X \to I$ is called a *smooth topology* (in short, st) on X if τ' satisfies the following axioms:

(i) $\tau'(\underline{0}) = \tau'(\underline{1}) = 1.$

(ii) $\forall A_1, A_2 \in I^X, \tau'(A_1 \cap A_2) \geq \tau'(A_1) \wedge \tau'(A_2).$

(iii) $\forall \Gamma, \tau'(\bigcup_{\alpha \in \Gamma} A_\alpha) \geq \bigwedge_{\alpha \in \Gamma} \tau'(A_\alpha).$

The pair (X, τ') is called a *smooth topological space* (in short, sts). We will denote the set of all st's on X as $ST(X)$.

Definition 2.2.3. [3] Let X be a nonempty set. Then a mapping $\mathcal{C}' : I^X \to I$ is called a *smooth cotopology* (in short, sct) on X subsets of X if $\mathcal{C}'$ satisfies the following axioms:

(i) $\mathcal{C}'(\underline{0}) = \mathcal{C}'(\underline{1}) = 1.$

(ii) $\forall B_1, B_2 \in I^X, \mathcal{C}'(B_1 \cup B_2) \geq \mathcal{C}'(B_1) \wedge \mathcal{C}'(B_2)$.

(iii) $\forall \Gamma, \mathcal{C}'(\bigcap_{\alpha\in\Gamma} B_\alpha) \geq \bigwedge_{\alpha\in\Gamma} \mathcal{C}'(B_\alpha)$.

The pair $(X, \mathcal{C}')$ is called a *smooth cotopological space* (in short, $scts$). We will denote the set of all sct's on X as $SCT(X)$.

Remark 2.2.4. If $I = \{0, 1\}$, then definition 2.2.2 coincides with the known definition of classical topology.

The operators on X which induced by the smooth topologies τ' are follows:

Definition 2.2.5. [3] Let (X, τ') be an osts and let $A \in 2^X$. Then *smooth closure* of A in X , denoted by $\overline{A}$ is defined by

$$\overline{A} = \begin{cases} A, & \text{if } \tau'(A^c) = 1, \\ \bigcap\{F \in 2^X : A \subseteq F \text{ and } \tau'(F^c) > \tau'(A^c)\}, & \text{if } \tau'(A^c) \neq 1. \end{cases}$$

Definition 2.2.6. [3] Let (X, τ') be an osts and let $A \in 2^X$. Then *smooth interior* of A in X , denoted by A° is defined by

$$A^\circ = \begin{cases} A, & \text{if } \tau'(A) = 1, \\ \bigcup\{S \in 2^X : S \subseteq A \text{ and } \tau'(S) > \tau'(A)\}, & \text{if } \tau'(A) \neq 1. \end{cases}$$

Definition 2.2.7. [3] A map $f : X \to Y$ is called a *smooth continuous* with respect to the smooth topologies τ'_1 and τ'_2 respectively, iff for every $A \in L^Y$ we have $\tau'_2(A) \leq \tau'_1(f^{-1}(A))$, where $f^{-1}(A)$ is defined by $f^{-1}(A)(x) = A(f(x)), \forall x \in X$.

For a smooth topological space (X, τ'), we define $suppA = \{x \in X : A(x) > 0\}$ and $suppA$ will be called the *support* of τ'.

Next, we will recall the definitions of ST_0, ST_1 and ST_2-spaces in smooth topological spaces.

Definition 2.2.8. [4] A sts (X, τ') is called ST_0- *space* if and only if for each $x, y \in X$ with $x \neq y$ there exists $A \in I^X$ such that ($x \in suppA, y \notin suppA$ and $\tau'(A) \geq A(x)$) or ($y \in suppA, x \notin suppA$ and $\tau'(A) \geq A(y)$).

Definition 2.2.9. [4] A sts (X, τ') is called ST_1- *space* if and only if for each $x, y \in X$ with $x \neq y$ there exist $A, B \in I^X$ such that ($x \in suppA \backslash suppB$ and $\tau'(A) \geq A(x)$) or ($y \in suppB \backslash suppA$ and $\tau'(B) \geq B(y)$).

Definition 2.2.10. [4] A sts (X, τ') is called ST_2- *space* if and only if for each $x, y \in X$ with $x \neq y$ there exist $A, B \in I^X$ such that $x \in suppA, \tau'(A) \geq A(x)$, $y \in suppB, \tau'(B) \geq B(y)$ and $A \cap B = \emptyset$.

2.3 Ordinary smooth topological spaces

In this section we discuss some properties of ordinary smooth topological spaces and some properties of smooth closure, smooth interior, smooth continuous, ordinary smooth open, ordinary smooth closed and ordinary smooth subspaces in ordinary smooth topological spaces.

For each a nonempty set X, let 2^X be the set of all ordinary subsets of a set X and let I be the closed interval [0, 1]. For any $\tau : 2^X \to I$, the infimum and the supremum of $\{\tau(A_\alpha) : \alpha \in \Gamma\}$ are defined as follows:

$$\bigwedge_{\alpha \in \Gamma} \tau(A_\alpha) = \inf\{\tau(A_\alpha) : \alpha \in \Gamma\}.$$

$$\bigvee_{\alpha \in \Gamma} \tau(A_\alpha) = \sup\{\tau(A_\alpha) : \alpha \in \Gamma\}.$$

Definition 2.3.1. [5] Let X be a nonempty set. Then a mapping $\tau : 2^X \to I$ is called an *ordinary smooth topology* (in short, ost) on X or a *gradation of openness of ordinary subsets of* X if τ satisfies the following axioms:

(i) $\tau(\emptyset) = \tau(X) = 1$.

(ii) $\tau(A \cap B) \geq \tau(A) \wedge \tau(B) \quad \forall A, B \in 2^X$.

(iii) $\tau(\bigcup_{\alpha \in \Gamma} A_\alpha) \geq \bigwedge_{\alpha \in \Gamma} \tau(A_\alpha) \quad \forall \{A_\alpha\} \subseteq 2^X$.

The pair (X, τ) is called an *ordinary smooth topological space* (in short, $osts$). We will denote the set of all ost on X as $OST(X)$.

Definition 2.3.2. [5] Let X be a nonempty set. Then a mapping $\mathcal{C} : 2^X \to I$ is called an *ordinary smooth cotopology* (in short, $osct$) on X or a gradation of closedness of ordinary subsets of X if $\mathcal{C}$ satisfies the following axioms:

(i) $\mathcal{C}(\emptyset) = \mathcal{C}(X) = 1$.

(ii) $\mathcal{C}(A \cup B) \geq \mathcal{C}(A) \wedge \mathcal{C}(B) \quad \forall A, B \in 2^X$.

(iii) $\mathcal{C}(\bigcap_{\alpha\in\Gamma} A_\alpha) \geq \bigwedge_{\alpha\in\Gamma} \mathcal{C}(A_\alpha) \quad \forall\{A_\alpha\} \subseteq 2^X$.

The pair (X, $\mathcal{C}$) is called an *ordinary smooth cotopological space* (in short, $oscts$). We will denote the set of all osct on X as $OSCT(X)$.

Remark 2.3.3. If $I = \{0, 1\}$, then Definition 2.3.1 coincides with the known definition of classical topology.

Proposition 2.3.4. [5] Let (X, τ) be an osts and let $A \subseteq X$. We defined a mapping $\tau_A : 2^A \to I$ as follows: For each $B \in 2^A$,

$$\tau_A(B) = \bigvee\{\tau(C) : C \in 2^X \text{ and } C \cap A = B\}.$$

Then $\tau_A \in OST(A)$ and $\tau(B) \leq \tau_A(B)$. In this case, (A, τ_A) is called an *ordinary smooth subspace* of (X, τ) and τ_A is called the *induced ordinary smooth topology* on A by τ.

The operators on X which induced by the ordinary smooth topologies τ are follows:

Definition 2.3.5. [6] Let (X, τ) be an osts and let $A \in 2^X$. Then *ordinary smooth closure* of A in X, denoted by $\overline{A}$ is defined by

$$\overline{A} = \bigcap\{F \in 2^X : A \subseteq F \text{ and } \mathcal{C}_\tau(F) > 0\}.$$

Definition 2.3.6. [6] Let (X, τ) be an osts and let $A \in 2^X$. Then *ordinary smooth interior* of A in X, denoted by A° is defined by

$$A^\circ = \bigcup\{U \in 2^X : U \subseteq A \text{ and } \tau(U) > 0\}.$$

The following results therefore follows directly from the definition of ordinary smooth closure and ordinary smooth interior.

Proposition 2.3.7. [6] Let (X, τ) be an osts and let $A, B \in 2^X$. Then:

(i) If $A \subseteq B$, then $A^\circ \subseteq B^\circ$ and $\overline{A} \subseteq \overline{B}$.

(ii) $(A^\circ)^c = \overline{(A^c)}$.

(iii) $A^\circ = (\overline{(A^c)})^c$.

(iv) $\overline{A} = ((A^c)^\circ)^c$.

(v) $(\overline{A})^c = (A^c)^\circ$.

Proposition 2.3.8. [6] Let (X, τ) be an osts and let $A, B \in 2^X$. Then:

(i) $X^\circ = X$.

(ii) $A^\circ \subseteq A$.

(iii) $(A^\circ)^\circ = A^\circ$.

(iv) $(A \cap B)^\circ \subseteq A^\circ \cap B^\circ$.

Proposition 2.3.9. [6] Let (X, τ) be an osts and let $A, B \in 2^X$. Then:

(i) $\overline{\emptyset} = \emptyset$.

(ii) $A \subseteq \overline{A}$.

(iii) $\overline{(\overline{A})} = \overline{A}$.

(iv) $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$.

Proposition 2.3.10. [6] Let (X, τ) be an osts and let $A, B \in 2^X$. Then:

(i) If $\tau(A) > 0$, then $A = A^\circ$.

(ii) If $\mathcal{C}_\tau(A) > 0$, then $A = \overline{A}$.

Definition 2.3.11. [6] Let (X, τ_1) and (Y, τ_2) be two osts. Then a mapping $f : X \to Y$ is said to be *ordinary smooth continuous* if $\tau_2(A) \leq \tau_1(f^{-1}(A)),\ \forall A \in 2^Y$.

Definition 2.3.12. [6] Let (X, τ_1) and (Y, τ_2) be two osts. Then a mapping $f : X \to Y$ is said to be *ordinary smooth continuous* if $\tau_2(A^c) \leq \tau_1(f^{-1}(A)^c),\ \forall A \in 2^Y$.

Corollary 2.3.13. [6] Let (X, τ_1) and (Y, τ_2) be two osts and let $f : X \to Y$ be ordinary smooth continuous. Then:

(i) $f(\overline{A}) \subseteq \overline{f(A)}$ for all $A \in 2^X$.

(ii) $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$ for all $B \in 2^Y$.

(iii) $f^{-1}(B^\circ) \subseteq (f^{-1}(B))^\circ$ for all $B \in 2^Y$.

Definition 2.3.14. [6] Let $\tau_1 \in OST(X)$ and let $\tau_2 \in OST(Y)$. Then a mapping $f : X \to Y$ is said to be

(i) *ordinary smooth open* if $\tau_1(A) \leq \tau_2(f(A)), \forall A \in 2^X$.

(ii) *ordinary smooth closed* if $\tau_1(A^c) \leq \tau_2(f(A^c)), \forall A \in 2^X$.

Definition 2.3.15. [5] Let $\tau_1 \in OST(X)$ and let $\tau_2 \in OST(Y)$. Then a mapping $f : X \to Y$ is called an *ordinary smooth homeomorphism* if :

(i) f is a bijective,

(ii) f and f^{-1} are ordinary smooth continuous.

Theorem 2.3.16. [5] Let (X, τ_1) and (Y, τ_2) be two osts's and let $f : X \to Y$ be a mapping. Then the following are equivalent:

(i) f is an ordinary smooth homeomorphism.

(ii) f is ordinary smooth open and ordinary smooth continuous.

(iii) f is ordinary smooth closed and ordinary smooth continuous.

CHAPTER 3

SEPARATION AXIOMS IN ORDINARY SMOOTH TOPOLOGICAL SPACES

In this chapter, we will introduce the notion of $OT_0, OT_1, OT_2, OT_3, OT_4$ and OT_5-spaces on ordinary smooth topological spaces and study some of their properties, by using $S(\tau)$ operator.

For an osts (X, τ), define $S(\tau) = \{A \in 2^X : \tau(A) > 0\}$ and $S(\tau)$ will be called the *support* of τ.

3.1 OT_0-*spaces*

In this subsection, we will introduce the notion of OT_0-*spaces* and investigate some of their properties.

Definition 3.1.1. An osts (X, τ) is called a OT_0-*space* if and only if for each $x, y \in X$ with $x \neq y$, there exists $U \in S(\tau)$ such that either $x \in U$ and $y \notin U$ or $y \in U$ and $x \notin U$.

Example 3.1.2. Let $X = \{1, 2\}$ and we define a mapping $\tau : 2^X \to I$ as follows: $\tau(X) = \tau(\emptyset) = 1, \tau(\{1\}) = 0.5, \tau(\{2\}) = 0$. Clearly, (X, τ) is an osts. Then (X, τ) is a OT_0-space, since $\{1\} \in S(\tau)$, $1 \in \{1\}$ and $2 \notin \{1\}$.

We now give an example of an osts which is not OT_0-spaces.

Example 3.1.3. Let $X = \{1, 2, 3\}$ and we define a mapping $\tau : 2^X \to I$ as follows: $\tau(X) = \tau(\emptyset) = 1, \tau(\{1\}) = 0.5$ and $\tau(A) = 0$ if $A \notin \{X, \emptyset, \{1\}\}$. Clearly, (X, τ) is an osts. Since X is the only set in $S(\tau)$ which contain 2 and 3. That means there are no $U \in S(\tau)$ such that $2 \in U$ and $3 \notin U$.

Next theorem shows a very simple characterization of OT_0-spaces.

Theorem 3.1.4. An osts (X, τ) is a OT_0-space if and only if for every $x, y \in X$ such that $x \neq y$, we have that $\overline{\{x\}} \neq \overline{\{y\}}$.

Proof. $(\Longrightarrow)$: Suppose that (X,τ) is an OT_0-space and let $x,y \in X$ such that $x \neq y$. By assumption we may assume that there exists $U \in S(\tau)$ such that $x \in U, y \notin U$. Sine $\{y\} \subseteq X\backslash U$ and $\tau(U) > 0$, using Proposition 2.3.10 (ii), we get, $\overline{\{y\}} \subseteq X\backslash U$. But $\{x\} \nsubseteq X\backslash U$. Therefore $\overline{\{x\}} \neq \overline{\{y\}}$.
$(\Longleftarrow)$: Assume that $\overline{\{x\}} \neq \overline{\{y\}}$ for all $x,y \in X$ such that $x \neq y$. We will show that (X,τ) is a OT_0-space. By assumption, we may assume that $x \notin \overline{\{y\}}$. Then there exists $F \in 2^X$ such that $\{y\} \subseteq F$, $\tau(F^c) > 0$ and $x \notin F$. Let $U = F^c$. Since $x \notin F$ and $y \in F$, then $x \in U$ and $y \notin U$. Therefore (X,τ) is a OT_0-space. □

Proposition 3.1.5. Every subspace of OT_0-spaces is also OT_0-spaces.

Proof. Let (X,τ) be an OT_0-space, let (A,τ_A) be an ordinary smooth subspace of (X,τ) and let a_1, a_2 be elements of A such that $a_1 \neq a_2$. Since (X,τ) is a OT_0-space, we may assume that there exists $U \in S(\tau)$ such that $a_1 \in U, a_2 \notin U$. Let $V = U \cap A$. Then

$$\begin{aligned}\tau_A(V) &= \bigvee\{\tau(U') : U' \in 2^X \text{ and } U' \cap A = V\}\\ &\geq \tau(U)\\ &> 0.\end{aligned}$$

So $V \in S(\tau_A)$ such that $a_1 \in V$ and $a_2 \notin V$. Hence (A,τ_A) is a OT_0-space. □

The following results are the properties of OT_0-spaces under some kinds of ordinary smooth maps.

Proposition 3.1.6. Let (X,τ_1) and (Y,τ_2) be two osts and let $f : X \to Y$ be an ordinary smooth homeomorphism. Then (X,τ_1) is an OT_0-space if and only if (Y,τ_2) is an OT_0-space.

Proof. $(\Longrightarrow)$: Let $y_1, y_2 \in Y$ such that $y_1 \neq y_2$. Since f is a bijective, then there are $x_1, x_2 \in X$ such that $y_1 = f(x_1), y_2 = f(x_2)$ and $x_1 \neq x_2$. Since (X,τ_1) is an OT_0-space, we may assume that there exists $U \in S(\tau_1)$ such that $x_1 \in U,\ x_2 \notin U$. Since f is an ordinary smooth open, it follows that

$$\tau_2(f(U)) \geq \tau_1(U) > 0.$$

Thus $f(U) \in S(\tau_2)$. Since f is an injective, then $y_1 \in f(U)$ and $y_2 \notin f(U)$. Hence (Y, τ_2) is a OT_0-space.

$(\Longleftarrow)$: Let $x_1, x_2 \in X$ such that $x_1 \neq x_2$. Since f is a bijective, then there are $y_1, y_2 \in Y$ such that $x_1 = f^{-1}(y_1), x_2 = f^{-1}(y_2)$ and $y_1 \neq y_2$. Since (Y, τ_2) is an OT_0-space, we may assume that there exists $U \in S(\tau_2)$ such that $y_1 \in U,\ y_2 \notin U$. Since f is an ordinary smooth continuous, it follows that

$$\tau_1(f^{-1}(U)) \geq \tau_2(U) > 0.$$

Thus $f^{-1}(U) \in S(\tau_1)$. Since f is an injective, then $x_1 \in f^{-1}(U)$ and $x_2 \notin f^{-1}(U)$. Hence (X, τ_1) is a OT_0-space.

□

Proposition 3.1.7. Let $f : X \to Y$ be injective, ordinary smooth continuous map with respect to the ordinary smooth topologies τ_1 and τ_2, respectively. If (Y, τ_2) is a OT_0-space, then so is (X, τ_1).

Proof. Let $x_1, x_2 \in X$ such that $x_1 \neq x_2$. Since f is an injective, we have $f(x_1) \neq f(x_2)$. Since (Y, τ_2) is a OT_0-space, we may assume that there exists $U \in S(\tau_2)$ such that $f(x_1) \in U,\ f(x_2) \notin U$. Since f is an injective and ordinary smooth continuous, it follows that

$$\tau_1(f^{-1}(U)) \geq \tau_2(U) > 0,$$

$$x_1 = f^{-1}(f(x_1)) \in f^{-1}(U),\ x_2 = f^{-1}(f(x_2)) \notin f^{-1}(U).$$

So, there exists $f^{-1}(U) \in S(\tau_1)$ such that $x_1 \in f^{-1}(U),\ x_2 \notin f^{-1}(U)$. Hence (X, τ_1) is a OT_0-space. □

Proposition 3.1.8. An osts (X, τ_1) is a OT_0-space. If $f : X \to Y$ is an injective and ordinary smooth open, then $(f(X), \tau_{2f(X)})$ is a OT_0-space.

Proof. Let $(f(X), \tau_{2f(X)})$ be an ordinary smooth subspace of (Y, τ_2). For any $a, b \in f(X)$ such that $a \neq b$. Since f is an injective, there exist $x, y \in X$ such that $x = f^{-1}(a) \neq f^{-1}(b) = y$. Since (X, τ_1) is a OT_0-space, we may assume that there exists $U \in S(\tau_1)$ such that $x \in U,\ y \notin U$. Since f is an ordinary smooth open and $f(U) \subseteq f(X)$, then

$$0 < \tau_1(U) \leq \tau_2(f(U)) \leq \tau_{2f(X)}(f(U)).$$

Thus $f(U) \in S(\tau_{2f(X)})$. Since f is an injective, then $a \in f(U),\ b \notin f(U)$. Therefore, $(f(X), \tau_{2f(X)})$ is a OT_0-space. □

Proposition 3.1.9. An osts (Y, τ_2) is a OT_0-space. If $f : X \to Y$ is an injective and ordinary smooth continuous, then $(f^{-1}(Y), \tau_{2f^{-1}(Y)})$ is a OT_0-space.

Proof. Let $(f^{-1}(Y), \tau_{1f^{-1}(Y)})$ be an ordinary smooth subspace of (X, τ_1). For any $a, b \in f^{-1}(Y)$ such that $a \neq b$, we have that $f(a) \neq f(b)$. Since (Y, τ_2) is a OT_0-space, we may assume that there exists $U \in S(\tau_2)$ such that $f(a) \in U,\ f(b) \notin U$. Since f is an ordinary smooth continuous and ordinary smooth subspace and $f^{-1}(U) \subseteq f^{-1}(Y)$, then

$$0 < \tau_2(U) \leq \tau_1(f^{-1}(U)) \leq \tau_{1f^{-1}(Y)}(f^{-1}(U)).$$

Thus $f^{-1}(U) \in S(\tau_{1f^{-1}(V)})$. Since f is an injective, then $a \in f^{-1}(U)$ and $b \notin f^{-1}(U)$. Hence $(f^{-1}(Y), \tau_{1f^{-1}(Y)})$ is a OT_0-space. □

3.2 OT_1-*spaces*

In this section, we will introduce the notion of OT_1-spaces and investigate some of their properties.

Definition 3.2.1. An osts (X, τ) is called a OT_1-*space* if and only if for each $x, y \in X$ with $x \neq y$, there exist $U, V \in S(\tau)$ such that $x \in U, y \notin U$ and $y \in V,\ x \notin V$.

Example 3.2.2. Let X be infinite set. We define a mapping $\tau : 2^X \to I$ as follows:

$$\tau(A) = \begin{cases} 1, & \text{if } A = \emptyset \text{ or } A^c \text{ is finite,} \\ 0, & \text{otherwise.} \end{cases}$$

for each $A \in 2^X$,

Clearly, (X, τ) is an osts. Let consider $x, y \in X$ such that $x \neq y$. Since $y \notin \{x\}$, then $y \in X\backslash\{x\}$. and $X\backslash\{x\} \in S(\tau)$. Similarly, $x \in X\backslash\{y\}$ and $X\backslash\{y\} \in S(\tau)$. It follows that (X, τ) is a OT_1-space.

Remark 3.2.3. If an osts (X, τ) is a OT_1-space, then (X, τ) is a OT_0-space.
Since (X, τ) is a OT_1-space and for any $x, y \in X$ which $x \neq y$, there exist $U, V \in S(\tau)$ such that $x \in U,\ y \notin U$ and $y \in V,\ x \notin V$. Hence, there exists $W \in S(\tau)$ such that $x \in W,\ y \notin W$ or $y \in W,\ x \notin W$. Therefore (X, τ) is a OT_0-space.

The converse of remark 3.2.3 is not true. If can be seen from the following example.

Example 3.2.4. From examlple 3.1.2, $(X, \tau\ )$ is not a OT_1-space. Since there exists $U \in S(\tau)$ which contain $1 \in U$ and $2 \notin U$. But there is not $V \in S(\tau)$ which $2 \in V$ but $1 \notin V$. That means there are not $U, V \in S(\tau)$ such that $2 \notin U$ and $1 \notin V$.

Theorem 3.2.5. An osts (X, τ) is a OT_1-space if and only $\overline{\{x\}} = \{x\}$ for every $x \in X$.

Proof. $(\Longrightarrow)$: Assume that (X, τ) is a OT_1-space. We will show that $\overline{\{x\}} = \{x\}$.
Suppose that $\overline{\{x\}} \backslash \{x\} \neq \emptyset$. Then there exists $y \in \overline{\{x\}} \backslash \{x\}$. Since (X, τ) is a OT_1-space, there exists $U_y \in S(\tau)$ such that $y \in U_y$ and $x \notin U_y$, so $y \notin (U_y)^c$ and $\{x\} \subseteq (U_y)^c$. But $y \in \overline{\{x\}}$, we have that $y \in F$ for all F such that $\{x\} \subseteq F$ and $F^c \in S(\tau)$. Since $\{x\} \subseteq (U_y)^c$ and $U_y \in S(\tau)$, then $y \in (U_y)^c$. This is a contradiction. Thus $\overline{\{x\}} \backslash \{x\} = \emptyset$. Hence $\overline{\{x\}} = \{x\}$.
$(\Longleftarrow)$: Assume that $\overline{\{x\}} = \{x\}$ for all $x \in X$. We will show that (X, τ) is a OT_1-space. Let $x, y \in X$ with $x \neq y$. By assumption we have that $x \in X \backslash \overline{\{y\}},\ y \in X \backslash \overline{\{x\}}$, then $x \notin \overline{\{y\}}$ and $y \notin \overline{\{x\}}$, there exist $F_1, F_2 \in 2^X$ such that $\{y\} \subseteq F_1,\ \{x\} \subseteq F_2$, $\tau(F_1^c) > 0,\ \tau(F_2^c) > 0$ and $x \notin F_1,\ y \notin F_2$. Let $U_1 = F_1^c$ and $U_2 = F_2^c$. Then $x \in U_1$, $x \notin U_2$ and $y \in U_2$, $y \notin U_1$, where $U_1, U_2 \in S(\tau)$. Therefore (X, τ) is a OT_1-space. □

Proposition 3.2.6. Every subspace of OT_1-spaces is also OT_1-spaces.

Proof. Let (X, τ) be a OT_1-space, let (A, τ_A) be an ordinary smooth subspace of (X, τ) and let a_1, a_2 be elements of A such that $a_1 \neq a_2$. Since(X, τ) is a OT_1-space, we may assume that there exist $U, V \in S(\tau)$ such that $a_1 \in U,\ a_2 \notin U$ and $a_2 \in V,\ a_1 \notin V$.
Let $B = U \cap A$ and $C = V \cap A$. Then

$$\begin{aligned}\tau_A(B) &= \bigvee\{\tau(U) : U \in 2^X \text{ and } U \cap A = B\} \\ &\geq \tau(U)\end{aligned}$$

$$> 0,$$

and

$$\begin{aligned}\tau_A(C) &= \bigvee\{\tau(V) : V \in 2^X \text{ and } V \cap A = B\}\\ &\geq \tau(V)\\ &> 0.\end{aligned}$$

So $B, C \in S(\tau_A)$ such that $a_1 \in B,\ a_2 \notin B$ and $a_2 \in C,\ a_1 \notin C$. Hence (A, τ_A) is a OT_1-space. □

The following results are the properties of OT_0-spaces under some kinds of ordinary smooth maps.

Proposition 3.2.7. Let (X, τ_1) and (Y, τ_2) be two osts and let $f : X \to Y$ be an ordinary smooth homeomorphism. Then (X, τ_2) is an OT_1-space if and only if (Y, τ_2) is an OT_1-space.

Proof. $(\Longrightarrow)$: Let $y_1, y_2 \in Y$ such that $y_1 \neq y_2$. Since f is a bijective, then there are $x_1, x_2 \in X$ such that $y_1 = f(x_1), y_2 = f(x_2)$ and $x_1 \neq x_2$. Since (X, τ_1) is a OT_1-space, then there exist $U, V \in S(\tau_1)$ such that $x_1 \in U$, $x_2 \notin U$ and $x_2 \in V$, $x_1 \notin V$. Since f is an ordinary smooth open, it follows that

$$\tau_2(f(U)) \geq \tau_1(U) > 0,$$

and

$$\tau_2(f(V)) \geq \tau_1(V) > 0.$$

Thus $f(U), f(V) \in S(\tau_2)$. Since f is an injective, then $y_1 \in f(U)$, $y_2 \notin f(U)$ and $y_2 \in f(V)$, $y_1 \notin f(V)$. Hence (Y, τ_2) is a OT_1-space.
$(\Longleftarrow)$: Let $x_1, x_2 \in X$ such that $x_1 \neq x_2$. Since f is a bijective, then there are $y_1, y_2 \in Y$ such that $x_1 = f^{-1}(y_1), x_2 = f^{-1}(y_2)$ and $y_1 \neq y_2$. Since (Y, τ_2) is a OT_1-space, then there exist $U, V \in S(\tau_2)$ such that $y_1 \in U$, $y_2 \notin U$ and $y_2 \in V$, $y_1 \notin V$. Since f is an ordinary smooth continuous, it follows that

$$\tau_1(f^{-1}(U)) \geq \tau_2(U) > 0,$$

and

$$\tau_1(f^{-1}(V)) \geq \tau_2(V) > 0.$$

Thus $f^{-1}(U), f^{-1}(V) \in S(\tau_1)$. Since f is an injective, then $x_1 \in f^{-1}(U)$, $x_2 \notin f^{-1}(U)$ and $x_2 \in f^{-1}(V)$, $x_1 \notin f^{-1}(V)$. Hence (X, τ_1) is a OT_1-space. □

Proposition 3.2.8. Let $f : X \to Y$ be an injective, ordinary smooth continuous map with respect to the ordinary smooth topologies τ_1 and τ_2 respectively. If (Y, τ_2) is a OT_1-space, then so is (X, τ_1).

Proof. Let $x_1, x_2 \in X$ with $x_1 \neq x_2$, since f is an injective, we have $f(x_1) \neq f(x_2)$. Since (Y, τ_2) is a OT_1-space, then there exist $U, V \in S(\tau_2)$ such that $f(x_1) \in U,\ f(x_2) \notin U$ and $f(x_2) \in V,\ f(x_1) \notin V$. Since f is an injective and ordinary smooth continuous, it follows that

$$\tau_1(f^{-1}(U)) \geq \tau_2(U) > 0,$$

$$\tau_1(f^{-1}(V)) \geq \tau_2(V) > 0,$$

$$x_1 = f^{-1}(f(x_1)) \in f^{-1}(U),\ x_2 = f^{-1}(f(x_2)) \notin f^{-1}(U)$$

and

$$x_2 = f^{-1}(f(x_2)) \in f^{-1}(V),\ x_1 = f^{-1}(f(x_1)) \notin f^{-1}(V).$$

So $f^{-1}(U), f^{-1}(V) \in S(\tau_1)$ such that $x_1 \in f^{-1}(U),\ x_2 \notin f^{-1}(U)$ and $x_2 \in f^{-1}(V),\ x_1 \notin f^{-1}(V)$. Hence (X, τ_1) is a OT_1-space. □

Proposition 3.2.9. An osts (X, τ_1) is a OT_1-space. If $f : X \to Y$ is an injective and ordinary smooth continuous, then $(f(X), \tau_{2f(X)})$ is a OT_1-space.

Proof. Let $(f(X), \tau_{2f(X)})$ be an ordinary smooth subspace of (Y, τ_2). For any $a, b \in f(X)$ such that $a \neq b$. Since f is an injective, then $f^{-1}(a) \neq f^{-1}(b)$. Since (X, τ_1) is a OT_1-space, then there exist $U, V \in S(\tau_1)$ such that $f^{-1}(a) \in U,\ f^{-1}(b) \notin U$ and $f^{-1}(b) \in V,\ f^{-1}(a) \notin V$. Since f is an ordinary smooth continuous and $(f(X), \tau_{2f(X)})$ is an ordinary smooth subspace of (Y, τ_2) and $f(U), f(V) \subseteq f(X)$, then

$$0 < \tau_1(U) \leq \tau_2(f(U)) \leq \tau_{2f(X)}(f(U)),$$

$$0 < \tau_1(V) \leq \tau_2(f(V)) \leq \tau_{2f(X)}(f(V)).$$

Thus $f(U), f(V) \in S(\tau_{2f(X)})$. Since f is an injective, then $a \in f(U),\ b \notin f(U)$ and $b \in f(V),\ a \notin f(V)$. Hence $(f(X), \tau_{2f(X)})$ is a OT_1-space. □

Proposition 3.2.10. An osts (Y, τ_2) is a OT_1-space. If $f : X \to Y$ is an injective and ordinary smooth open, then $(f^{-1}(Y), \tau_{1f^{-1}(Y)})$ is a OT_1-space.

Proof. Let $(f^{-1}(Y), \tau_{1f^{-1}(Y)})$ be an ordinary smooth subspace of (X, τ_1). For any $a, b \in f^{-1}(Y)$ such that $a \neq b$. Since f is an injective, then $f(a) \neq f(b)$. Since (Y, τ_2) is a OT_1-space, then there exist $U, V \in S(\tau_2)$ such that $f(a) \in U,\ f(b) \notin U$ and $f(b) \in V,\ f(a) \notin V$. Since f is an ordinary smooth open and $(f^{-1}(Y), \tau_{1f^{-1}(Y)})$ is an ordinary smooth subspace of (X, τ_1) and $f^{-1}(U), f^{-1}(V) \subseteq X$, then

$$\tau_{1f^{-1}(Y)}(f^{-1}(U)) \geq \tau_1(f^{-1}(U)) \geq \tau_2(U) > 0,$$

$$\tau_{1f^{-1}(Y)}(f^{-1}(V)) \geq \tau_1(f^{-1}(V)) \geq \tau_2(V) > 0.$$

Thus $f^{-1}(U), f^{-1}(V) \in S(\tau_{1f^{-1}(Y)})$ Since f is an injective, then $a \in f^{-1}(U),\ b \notin f^{-1}(U)$ and $b \in f^{-1}(V),\ a \notin f^{-1}(V)$. Hence $(f^{-1}(Y), \tau_{1f^{-1}(Y)})$ is a OT_1-space. □

3.3 OT_2-*spaces*

In this section, we will introduce the notion of OT_2-spaces and investigate some of their properties.

Definition 3.3.1. An osts (X, τ) is called a OT_2-*space* if and only if for each $x, y \in X$ with $x \neq y$, there exist $U, V \in S(\tau)$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

Example 3.3.2. Let X be a nonempty set. We define a mapping $\tau : 2^X \to I$ as follows:

$$\tau(A) = 1,$$

for each $A \in 2^X$.

Then pair (X, τ) is called an *ordinary smooth discrete topological space* on X. For each $x, y \in X$ which $x \neq y$. Since $\tau(\{x\}) = 1, \tau(\{y\}) = 1$, Then $\{x\}, \{y\} \in S(\tau)$ and $\{x\} \cap \{y\} = \emptyset$. Therefore (X, τ) is a OT_2-space.

Remark 3.3.3. If an osts (X, τ) is a OT_2-space, then (X, τ) is a OT_1-space.
Since (X, τ) is a OT_2-space and $x, y \in X$ with $x \neq y$, there exist $U, V \in S(\tau)$ such that $x \in U,\ y \in V$ and $U \cap V = \emptyset$. Hence $x \in U,\ y \notin U$ and $y \in V,\ x \notin V$. Therefore (X, τ) is a OT_1-space.

Remark 3.3.4. If an osts (X, τ) is a OT_1-space, then (X, τ) is not a OT_2-space, be seen from the following example.

Example 3.3.5. By example 3.2.2. Let X be a infinite set. We define a mapping $\tau : 2^X \to I$ as follows:

$$\tau(A) = \begin{cases} 1, & \text{if } A = \emptyset \text{ or } A^c \text{ is finite,} \\ 0, & \text{otherwise,} \end{cases}$$

for each $A \in 2^X$
Clearly, (X, τ) is a OT_1-space. But is not (X, τ) is a OT_2-space. It is enough to prove that $U \cap V \neq \emptyset$ for all $U, V \in S(\tau)$ and $U, V \neq \emptyset$. By the definition of τ, we have that $U = X \backslash \{x_1, x_2, ..., x_n\}$ and $V = X \backslash \{y_1, y_2, ..., y_m\}$ for some $x_1, x_2, ..., x_n, y_1, y_2, ..., y_m$ and $m, n \in \mathbb{N}$. Then

$$\begin{aligned} U \cap V &= (X \backslash \{x_1, x_2, ..., x_n\}) \cap (X \backslash \{y_1, y_2, ..., y_m\}) \\ &= X \backslash (\{x_1, x_2, ..., x_n\}) \cup \{y_1, y_2, ..., y_m\}) \\ &\neq \emptyset. \end{aligned}$$

Hence (X, τ) is not a OT_2-space.

Proposition 3.3.6. Every subspace of OT_2-spaces is also OT_2-spaces.

Proof. Let (X, τ) be an OT_2-space, let (A, τ_A) be an ordinary smooth subspace of (X, τ). For any $a_1, a_2 \in A$ such that $a_1 \neq a_2$. Since (X, τ) is a OT_2-space, then there exist $U, V \in S(\tau)$ such that $a_1 \in U, a_2 \in V$ and $U \cap V = \emptyset$. Let $B = U \cap A$ and $C = V \cap A$. Then

$$\begin{aligned} \tau_A(B) &= \bigvee \{\tau(U) : U \in 2^X \text{ and } U \cap A = B\} \\ &\geq \tau(U) \\ &> 0, \end{aligned}$$

$$\tau_A(C) = \bigvee\{\tau(V) : V \in 2^X \text{ and } V \cap A = C\}$$
$$\geq \tau(V)$$
$$> 0,$$

and

$$B \cap C = (U \cap A) \cap (V \cap A)$$
$$= (U \cap V) \cap A$$
$$= \emptyset \cap A$$
$$= \emptyset.$$

So $B, C \in S(\tau_A)$ such that $a_1 \in B$, $a_2 \in C$ and $B \cap C = \emptyset$. Hence (A, τ_A) is a OT_2-space. □

The next theorem, we give the equivalent conditions to be OT_2-spaces.

Theorem 3.3.7. Let (X, τ) be an osts. Then the following conditions are equivalent:

(i) (X, τ) is a OT_2-space.

(ii) Let $p \in X$ for $q \neq p$ there exists $U \in S(\tau), p \in U$ such that $q \notin \overline{U}$.

(iii) For each $p \in X$, $\cap\{\overline{U} : U \in S(\tau), p \in U\} = \{p\}$.

Proof. (i) ⇒ (ii) Let $p, q \in X$ with $q \neq p$. Since (X, τ) is a OT_2-space, there exist $U, V \in S(\tau)$ such that $p \in U, q \in V$ and $U \cap V = \emptyset$, then $U \subseteq X \backslash V$. By Proposition 2.3.7 (i), we have $\overline{U} \subseteq \overline{X \backslash V}$ and by Proposition 2.3.10 (ii), then $\overline{X \backslash V} = X \backslash V$. Since $q \in V$, then $q \notin X \backslash V$. Hence $q \notin \overline{U}$.
(ii) ⇒ (iii) Let $p \in X$. We will show that $\cap\{\overline{U} : U \in S(\tau), p \in U\} = \{p\}$. Clearly, $\{p\} \subseteq \cap\{\overline{U} : U \in S(\tau), p \in U\}$. Sufficient to proof that $\cap\{\overline{U} : U \in S(\tau), p \in U\} \subseteq \{p\}$. Let $q \in X, q \neq p$, then $q \notin \{p\}$. By (ii), then there exists $U_1 \in S(\tau), p \in U_1$ such that $q \notin \overline{U_1}$. Then $q \notin \cap\{\overline{U} : U \in S(\tau), p \in U\}$. Hence $\cap\{\overline{U} : U \in S(\tau), p \in U\} \subseteq \{p\}$. Therefore $\cap\{\overline{U} : U \in S(\tau), p \in U\} = \{p\}$.
(iii)⇒ (i) Assume that $\{p\} = \cap\{\overline{U} : U \in S(\tau), p \in U\}$. Let $p, q \in X$ with $p \neq q$, then $q \notin \{p\} = \cap\{\overline{U} : U \in S(\tau), p \in U\}$. Then there exists $U_1 \in S(\tau)$ such that $p \in U_1$ and

$q \notin \overline{U_1}$. Since $\overline{U_1} = \cap\{V \in 2^X : U_1 \subseteq V$ and $\tau(V^c) > 0\}$, then there exists $V \in 2^X$ such that $U_1 \subseteq V,\ \tau(V^c) > 0$ and $q \notin V$. Let $K = V^c$. Since $q \notin V$, then $q \in V^c = K$. Next, we will show that $K \in S(\tau)$. Since $K = V^c$, then $\tau(K) = \tau(V^c) > 0$. Hence $K \in S(\tau)$. Next, to show that $K \cap U_1 = \emptyset$. Since $V \cap V^c = \emptyset$ and $U_1 \subseteq V$. Then $K \cap U_1 = \emptyset$. Hence (X, τ) is a OT_2-space. □

The following results are the properties of OT_2-spaces under some kinds of ordinary smooth maps.

Proposition 3.3.8. Let (X, τ_1) and (Y, τ_2) be two osts and let $f : X \to Y$ be an ordinary smooth homeomorphism. Then (X, τ_2) is an OT_2-space if and only if (Y, τ_2) is an OT_2-space.

Proof. $(\Longrightarrow)$: Let $y_1, y_2 \in Y$ such that $y_1 \neq y_2$. Since f is a bijective, then there are $x_1, x_2 \in X$ such that $y_1 = f(x_1), y_2 = f(x_2)$ and $x_1 \neq x_2$. Since (X, τ_1) is a OT_2-space, then there exist $U, V \in S(\tau_1)$ such that $x_1 \in U, x_2 \in V$ and $U \cap V = \emptyset$. Since f is an ordinary smooth open, it follows that

$$\tau_2(f(U)) \geq \tau_1(U) > 0,$$

and

$$\tau_2(f(V)) \geq \tau_1(V) > 0.$$

Thus $f(U), f(V) \in S(\tau_2)$. Since f is a bijective, then $y_1 \in f(U), y_2 \in f(V)$ and $f(U) \cap f(V) = \emptyset$. Hence (Y, τ_2) is a OT_2-space.
$(\Longleftarrow)$: Let $x_1, x_2 \in X$ such that $x_1 \neq x_2$. Since f is a bijective, then there are $y_1, y_2 \in Y$ such that $x_1 = f^{-1}(y_1), x_2 = f^{-1}(y_2)$ and $y_1 \neq y_2$. Since (Y, τ_2) is a OT_2-space, then exist $U, V \in S(\tau_2)$ such that $y_1 \in U, y_2 \in V$ and $U \cap V = \emptyset$. Since f is an ordinary smooth continuous, it follows that

$$\tau_1(f^{-1}(U)) \geq \tau_2(U) > 0,$$

and

$$\tau_1(f^{-1}(V)) \geq \tau_2(V) > 0.$$

Thus $f^{-1}(U), f^{-1}(V) \in S(\tau_1)$. Since f is a bijective, then $x_1 \in f^{-1}(U), x_2 \in f^{-1}(V)$ and $f^{-1}(U) \cap f^{-1}(V) = \emptyset$. Hence (X, τ_1) is a OT_2-space. □

Proposition 3.3.9. Let $f : X \to Y$ be an injective, ordinary smooth continuous map with respect to the ordinary smooth topologies τ_1 and τ_2 respectively. If (Y, τ_2) is a OT_2-space, then so is a (X, τ_1).

Proof. Let $x_1, x_2 \in X$ such that $x_1 \neq x_2$. Since f is an injective, we have $f(x_1) \neq f(x_2)$. Since (Y, τ_2) is a OT_2-space, then there exist $U, V \in S(\tau_2)$ such that $f(x_1) \in U, f(x_2) \in V$ and $U \cap V = \emptyset$. Since f is an injective and ordinary smooth continuous, it follows that

$$\tau_1(f^{-1}(U)) \geq \tau_2(U) > 0,$$

$$\tau_1(f^{-1}(V)) \geq \tau_2(V) > 0,$$

$$f^{-1}(U) \cap f^{-1}(V) = \emptyset,$$

$$x_1 = f^{-1}(f(x_1)) \in f^{-1}(U),$$

and

$$x_2 = f^{-1}(f(x_2)) \in f^{-1}(V).$$

So $f^{-1}(U), f^{-1}(V) \in S(\tau_1)$ such that $x_1 \in f^{-1}(U), x_2 \in f^{-1}(V)$ and $f^{-1}(U) \cap f^{-1}(V) = \emptyset$. Hence (X, τ_1) is a OT_2-space. □

Proposition 3.3.10. An osts (X, τ) is a OT_2-space. If $f : X \to Y$ is an injective and ordinary smooth open, then $(f(X), \tau_{2f(X)})$ is a OT_2-space.

Proof. Let $(f(X), \tau_{2f(X)})$ be an ordinary smooth subspace of (Y, τ_2). For any $a, b \in f(X)$ such that $a \neq b$. Since f is an injective, then $f^{-1}(a) \neq f^{-1}(b)$. Since (X, τ_1) is a OT_2-space, then there exist $U, V \in S(\tau_1)$ such that $f^{-1}(a) \in U, f^{-1}(b) \in V$ and $U \cap V = \emptyset$. Since f is an ordinary smooth open and $(f(X), \tau_{2f(X)})$ is an ordinary smooth subspace of (Y, τ_2) and $f(U), f(V) \subseteq f(X)$, then

$$0 < \tau_1(U) \leq \tau_2(f(U)) \leq \tau_{2f(X)}(f(U)),$$

$$0 < \tau_1(V) \leq \tau_2(f(V)) \leq \tau_{2f(X)}(f(V)).$$

Thus $f(U), f(V) \in S(tau_{2f(X)})$.Since f is an injective, then $a \in f(U), b \in f(V)$ and $f(U) \cap f(V) = \emptyset$. Hence, $(f(X), \tau_{2f(X)})$ is a OT_2-space. □

Proposition 3.3.11. An osts (Y, τ_2) is a OT_2-space. If $f : X \to Y$ is an injective and ordinary smooth continuous, then $(f^{-1}(Y), \tau_{1f^{-1}(Y)})$ is a OT_2-space.

Proof. Let $(f^{-1}(Y), \tau_{1f^{-1}(Y)})$ be an ordinary smooth subspace of (X, τ_1). For any $a, b \in f^{-1}(Y)$ such that $a \neq b$. Since f is an injective, then $f(a) \neq f(b)$. Since (Y, τ_2) is a OT_2-space, then there exist $U, V \in S(\tau_2)$ such that $f(a) \in U, f(b) \in V$ and $U \cap V = \emptyset$. Since f is an ordinary smooth continuous and $(f^{-1}(Y), \tau_{1f^{-1}(Y)})$ is an ordinary smooth subspace of (X, τ_1) and $f^{-1}(U), f^{-1}(V) \subseteq X$, then

$$\tau_{1f^{-1}(Y)}(f^{-1}(U)) \geq \tau_1(f^{-1}(U)) \geq \tau_2(U) > 0,$$

$$\tau_{1f^{-1}(Y)}(f^{-1}(V)) \geq \tau_1(f^{-1}(V)) \geq \tau_2(V) > 0.$$

Thus $f^{-1}(U), f^{-1}(V) \in S(\tau_{1f^{-1}(Y)})$. Since f is an injective, then $a \in f^{-1}(U), b \in f^{-1}(V)$ and $f^{-1}(U) \cap f^{-1}(V) = \emptyset$. Hence $(f^{-1}(Y), \tau_{1f^{-1}(Y)})$ is a OT_2-space. □

3.4 OT_3-*spaces*

In this section, we will introduce the notion of OT_3-spaces and investigate some of their properties.

Definition 3.4.1. An osts (X, τ) is called a OT_3-*space* if and only if for each $A \subseteq X$, satisfying $\tau(A^c) > 0$, and each $b \in X$, satisfying $b \notin A$, there exist $U, V \in S(\tau)$ such that $A \subseteq U$, $b \in V$ and $U \cap V = \emptyset$.

Example 3.4.2. Let $X = \{a, b, c\}$ and we define the mapping $\tau : 2^X \to I$ as follows: $\tau(X) = \tau(\emptyset) = 1, \tau(\{a\}) = 0.6, \tau(\{b, c\}) = 0.4$ and $\tau(A) = 0$ if $A \notin \{X, \emptyset, \{a\}, \{b, c\}\}$. Clearly, (X, τ) is an osts. Since $\{a\}, \{b, c\} \in S(\tau)$ such that $\{a\} \cap \{b, c\} = \emptyset$. Hence $(X, \tau$ $)$ is a OT_3-space. But $(X, \tau$ $)$ is a OT_2-space, because b, c are not disjoint.

We now give an example of an osts which is not a OT_3-space.

Example 3.4.3. Let $X = \{a, b\}$ and we define a mapping $\tau : 2^X \to I$ as follows: $\tau(X) = \tau(\emptyset) = 1, \tau(\{a\}) = 0.9, \tau(\{b\}) = 0$. Clearly, (X, τ) is an osts. Since X is the only set in $S(\tau)$ which contains $\{a, c\}$, that means there are no $U \in S(\tau)$ such that $X \cap U \neq \emptyset$. Hence (X, τ) is not a OT_3-space.

Definition 3.4.4. An osts (X,τ) is said to be a *OT-regular space* if (X,τ) is a OT_3-space and OT_1-space.

Theorem 3.4.5. Let an osts (X,τ) is a OT-regular space, then (X,τ) is a OT_2-space.

Proof. Let (X,τ) is a OT-regular space. Then (X,τ) is a OT_3-space and OT_1-space. We will show that (X,τ) is a OT_2-space. Let $x,y \in X$ such that $x \neq y$. Since (X,τ) is a OT_1-space, there exist $U,V \in S(\tau)$ such that $x \in U,\ y \notin U$ and $y \in V,\ x \notin V$. Since $x \in U$, then $x \notin X\backslash U$ and

$$\tau((X\backslash U)^c) = \tau(U) > 0.$$

Since (X,τ) is a OT_3-space, there exist $W,Z \in S(\tau)$ such that $X\backslash U \subseteq W$, $x \in Z$ and $W \cap Z = \emptyset$. Since $y \notin U$, then $y \in X\backslash U \subseteq W$ and $x \in Z$ and $W \cap Z = \emptyset$. Therefore (X,τ) is a OT_2-space. □

Theorem 3.4.6. Every subspace of OT_3-spaces is also OT_3-spaces.

Proof. Let osts (X,τ) be a OT_3-space and let (A,τ_A) be an ordinary smooth subspace of (X,τ). For any $B \subseteq A$, satisfying $\tau_A(B^{c_A}) > 0$, and each $a \in A$, satisfying $a \notin B$. Since

$$\tau_A(B^{c_A}) = \bigvee\{\tau(C) : C \in 2^X \text{ and } C \cap A = B^{c_A}\}.$$

Then $C \cap A = B^{c_A}$, so $A \cap B^{c_A} \subseteq C$ and $B \cap C = \emptyset$. Let $\tau_A(B^{c_A}) = \delta$. Since $\delta = \bigvee\{\tau(C) : C \in 2^X \text{ and } C \cap A = B^{c_A}\}$, then there exists $C' \subseteq 2^X, C' \cap A = B^{c_A}$ such that

$$\tau(C') > \tau_A(B^{c_A}) - \frac{\delta}{2} = \delta - \frac{\delta}{2} = \frac{\delta}{2} > 0.$$

Since (X,τ) is a OT_3-space, there exist $U,V \in S(\tau)$ such that $(C')^c \subseteq U,\ a \in V$ and $U \cap V = \emptyset$. Let $E = U \cap A$ and $F = V \cap A$,

$$\begin{aligned}\tau_A(E) &= \bigvee\{\tau(U) : U \in 2^X \text{ and } U \cap A = E\}\\ &\geq \tau(U)\\ &> 0\end{aligned}$$

$$\begin{aligned}\tau_A(F) &= \bigvee\{\tau(V) : V \in 2^X \text{ and } V \cap A = E\}\\ &\geq \tau(V)\\ &> 0\end{aligned}$$

and

$$\begin{aligned} E \cap F &= (U \cap A) \cap (V \cap A) \\ &= (U \cap V) \cap A \\ &= \emptyset \cap A \\ &= \emptyset. \end{aligned}$$

Then $E, F \in S(\tau_A)$ such that $B \subseteq E, a \in F$ and $E \cap F = \emptyset$. Hence (A, τ_A) is a OT_3-space. □

The next theorem, we give the equivalent conditions to be OT_3-spaces.

Theorem 3.4.7. Let (X, τ) be an osts. Then the following conditions are equivalent:

(i) (X, τ) is a OT_3-space.

(ii) For each $x \in X$ and each U containing x, satisfying $\tau(U) > 0$, there exists a set V containing x, satisfying $\tau(V) > 0$ such that $x \in V \subseteq \overline{V} \subseteq U$.

(iii) For each $x \in X$, and each A not containing x, satisfying $\tau(A^c) > 0$, there exists a set V, satisfying $\tau(V) > 0$ containing x such that $\overline{V} \cap A = \emptyset$.

Proof. (i) $\Rightarrow$ (ii) Let $x \in X$ and each U containing x, satisfying $\tau(U) > 0$, then $x \notin X \backslash U$ and

$$\tau((X \backslash U)^c) = \tau(U) > 0.$$

Since (X, τ) is a OT_3-space, there exist $V, W \in S(\tau)$ such that $x \in V,\ X \backslash U \subseteq W$ and $V \cap W = \emptyset$, then $V \subseteq X \backslash W$. By Proposition 2.3.7 (i), then $\overline{V} \subseteq \overline{X \backslash W}$ and by Proposition 2.3.10 (ii), then $\overline{X \backslash W} = X \backslash W$. So $\overline{V} \subseteq X \backslash W$, but $X \backslash W \subseteq U$. Hence $\overline{V} \subseteq U$. Therefore $x \in V \subseteq \overline{V} \subseteq U$.

(ii) $\Rightarrow$ (iii) Let $x \in X$ and $A \subseteq X$, satisfying $\tau(A^c) > 0, x \notin A$, then $x \in X \backslash A$. By (ii), then there exists V, satisfying $\tau(V) > 0, x \in V$ such that $x \in V \subseteq \overline{V} \subseteq X \backslash A$. Since $A \cap (X \backslash A) = \emptyset$, hence $\overline{V} \cap A = \emptyset$.

(iii) $\Rightarrow$ (i) Assume that $A \subseteq X$, satisfying $\tau(A^c) > 0$, and let $b \in X$, satisfying $b \notin A$. By (iii), then there exists V, satisfying $\tau(V) > 0$ such that $\overline{V} \cap A = \emptyset$, $b \in V$. Since $\overline{V} \cap A = \emptyset$ and $\overline{V} = \cap\{F \in 2^X : V \subseteq F$ and $\tau(F^c) > 0\}$, then there exist $F \in 2^X$ such that $V \subseteq F,\ \tau(F^c) > 0$ and $A \cap F = \emptyset$. Hence $A \subseteq F^c$, $b \in V$ and $F^c, V \in S(\tau)$. Since $V \subseteq F$ and $F \cap F^c = \emptyset$, then $V \cap F^c = \emptyset$. Hence (X, τ) is a OT_3-space. □

The following results are the properties of OT_3-spaces under some kinds of ordinary smooth maps.

Proposition 3.4.8. Let (X, τ_1) and (Y, τ_2) be two osts and let $f : X \to Y$ be an ordinary smooth homeomorphism. Then (X, τ_2) is an OT_3-space if and only if (Y, τ_2) is an OT_3-space.

Proof. $(\Longrightarrow)$: Let $f : X \to Y$ be an ordinary smooth homeomorphism and let $A \subseteq Y$, satisfying $(\tau_2(A^c) > 0)$ and $b \in Y$, satisfying $b \notin A$. Since f is a bijective and ordinary smooth continuous, then $f^{-1}(b) \notin f^{-1}(A)$ and

$$\tau_1(f^{-1}(A)^c) = \tau_1(f^{-1}(A^c)) \geq \tau_2(A^c) > 0.$$

Since (X, τ_1) is a OT_3-space, then there exist $U, V \in S(\tau_1)$ such that $f^{-1}(A) \subseteq U,\ f^{-1}(b) \in V$ and $U \cap V = \emptyset$. Since f is an ordinary smooth open, it follows that

$$0 < \tau_1(U) \leq \tau_2(f(U)) \text{ and } 0 < \tau_1(V) \leq \tau_2(f(V)).$$

Thus $f(U), f(V) \in S(\tau_2)$ since f is an injective, then $A \subseteq f(U), b \in f(V)$ and $f(U) \cap f(V) = \emptyset$. Hence (Y, τ_2) is a OT_3-space.
$(\Longleftarrow)$: Let $A \subseteq X$, satisfying $(\tau_1(A^c) > 0)$ and $b \in X$, satisfying $b \notin A$. Since f is a bijective, then $f(b) \notin f(A)$. Since f is an ordinary smooth closed

$$\tau_2(f(A)^c) = \tau_2(f(A^c)) \geq \tau_1(A^c) > 0.$$

Since (Y, τ_2) is a OT_3-space, then there exist $U, V \in S(\tau_2)$ such that $f(A) \subseteq U,\ f(b) \in V$ and $U \cap V = \emptyset$. Since f is an ordinary smooth continuous, it follows that

$$0 < \tau_2(U) \leq \tau_1(f^{-1}(U)) \text{ and } 0 < \tau_2(V) \leq \tau_1(f^{-1}(V)).$$

Thus $f^{-1}(U), f^{-1}(V) \in S(\tau_1)$. Since f is an injective, then $A \subseteq f^{-1}(U), b \in f^{-1}(V)$ and $f^{-1}(U) \cap f^{-1}(V) = \emptyset$. Hence (X, τ_1) is a OT_3-space.

□

Proposition 3.4.9. Let $f : X \to Y$ be injective, ordinary smooth closed and ordinary smooth continuous map with respect to the ordinary smooth topologies τ_1 and τ_2 respectively. If (Y, τ_2) is a OT_3-space, then so is a (X, τ_1).

Proof. Let $A \subseteq X$, satisfying $\tau_1(A^c) > 0$ and $b \in X$ satisfying $b \notin A$. Since f is an injective and ordinary smooth closed, then $f(b) \notin f(A)$ and

$$\tau_2(f(A)^c) = \tau_2(f(A^c)) \geq \tau_1(A^c) > 0.$$

Since (Y, τ_2) is a OT_3-space, then there exist $U, V \in S(\tau_2)$ such that $f(A) \subseteq U, f(b) \in V$ and $U \cap V = \emptyset$. Since f is an injective and ordinary smooth continuous, it follows that

$$\tau_1(f^{-1}(U)) \geq \tau_2(U) > 0 \text{ and } \tau_1(f^{-1}(V)) \geq \tau_2(V) > 0,$$

$$f^{-1}(U) \cap f^{-1}(V) = \emptyset,$$

$$A = f^{-1}(f(A)) \subseteq f^{-1}(U),$$

and

$$b = f^{-1}(f(b)) \in f^{-1}(V).$$

So $f^{-1}(U), f^{-1}(V) \in S(\tau_1)$ such that $A \subseteq f^{-1}(U), b \in f^{-1}(V)$ and $f^{-1}(U) \cap f^{-1}(V) = \emptyset$. Hence (X, τ_1) is a OT_3-space. □

Proposition 3.4.10. An osts (X, τ_1) is a OT_3-space. If $f : X \to Y$ is an injective,ordinary smooth continuous and ordinary smooth open, then $(f(X), \tau_{2f(X)})$ is a OT_3-space.

Proof. Let $a \in f(X)$ and $A \subseteq f(X)$, satisfying $\tau_{2f(X)}(A^{c_{f(X)}}) > 0$ and $a \notin A$. Since

$$\tau_{2f(X)}(A^{c_{f(X)}}) = \bigvee\{\tau_2(C) : C \in 2^Y \text{ and } C \cap f(X) = A^{c_{f(X)}}\},$$

then $C \cap f(X) = A^{c_{f(X)}}$. Hence $f(X) \cap A^{c_{f(X)}} \subseteq C$ and $A \cap C = \emptyset$. Let $\tau_{2f(X)}(A^{c_{f(X)}}) = \delta$. Then $\delta = \bigvee\{\tau_2(C) : C \in 2^Y$ and $C \cap f(X) = A^{c_{f(X)}}\}$. Thus there exist $C' \subseteq 2^Y, C' \cap f(X) = A^{c_{f(X)}}$ such that

$$\begin{aligned}
\tau_2(C') &> \tau_{2f(X)}(A^{c_{f(X)}}) - \frac{\delta}{2} \\
&= \delta - \frac{\delta}{2} \\
&= \frac{\delta}{2} \\
&> 0.
\end{aligned}$$

Since f is ordinary smooth continuous we have

$$\tau_1(f^{-1}(C')) \geq \tau_2((C')) > 0.$$

And since $a \notin A$, then $a \in A^{c_{f(X)}}$. But $A^{c_{f(X)}} = C' \cap f(X)$. Thus $a \in C'$ and $a \in f(X)$. Hence $a \notin (C')^c$. Since f is an injective, then $f^{-1}(a) \notin f^{-1}((C')^c) = (f^{-1}(C'))^c$. Since (X, τ_1) is a OT_3-space, then there exist $U, V \in S(\tau_1)$ such that $f^{-1}((C')^c) \subseteq U, a \in V$ and $U \cap V = \emptyset$. Since f is an ordinary smooth open and $(f(X), \tau_{2f(X)})$ is an ordinary smooth subspace of (Y, τ_2) and $f(U), f(V) \subseteq f(X)$, then

$$0 < \tau_1(U) \leq \tau_2(f(U)) \leq \tau_{2f(X)}(f(U)),$$

$$0 < \tau_1(V) \leq \tau_2(f(V)) \leq \tau_{2f(X)}(f(V)),$$

Thus $f(U), (f(V)) \in S(\tau_{2f(X)})$. Since f is an injective, then $a \in f(V)$, $(C')^c = f(f^{-1}((C')^c))) \subseteq f(U)$ and $f(U) \cap f(V) = \emptyset$. Hence $(f(X), \tau_{2f(X)})$ is a OT_3-space. □

Proposition 3.4.11. An osts (Y, τ_2) is a OT_3-space. If $f : X \to Y$ is an injective, ordinary smooth closed and ordinary smooth open, then $(f^{-1}(Y), \tau_{2f^{-1}(Y)})$ is a OT_3-space.

Proof. Let $a \in f^{-1}(Y)$ and $A \subseteq f^{-1}(Y)$, satisfying $\tau_{1f^{-1}(Y)}(A^{c_{f^{-1}(Y)}}) > 0$ and $a \notin A$. Since f is an injective, then $f(a) \notin f(A)$. Since

$$\tau_{1f^{-1}(Y)}(A^{c_{f^{-1}(Y)}}) = \bigvee\{\tau_1(C) : C \in 2^X \text{ and } C \cap f^{-1}(Y) = A^{c_{f^{-1}(Y)}}\}.$$

Then $C \cap f^{-1}(Y) = A^{c_{f^{-1}(Y)}}$. Hence $f^{-1}(Y) \cap A^{c_{f^{-1}(Y)}} \subseteq C$ and $A \cap C = \emptyset$. Let $\tau_{1f^{-1}(Y)}(A^{c_{f^{-1}(Y)}}) = \delta$. Then $\delta = \bigvee\{\tau_1(C) : C \in 2^X$ and $C \cap f^{-1}(Y) = A^{c_{f^{-1}(Y)}}\}$. Thus there exist $C' \subseteq 2^X, C' \cap f^{-1}(Y) = A^{c_{f^{-1}(Y)}}$ such that

$$\begin{aligned}\tau_1(C') &> \tau_{1f^{-1}(Y)}(A^{c_{f^{-1}(Y)}}) - \frac{\delta}{2}\\ &= \delta - \frac{\delta}{2}\\ &= \frac{\delta}{2}\\ &> 0.\end{aligned}$$

Since f is an ordinary smooth open we have

$$\tau_2(f(C')) \geq \tau_1(C') > 0.$$

And since $a \notin A$, then $a \in A^{c_{f^{-1}(Y)}}$. But $A^{c_{f^{-1}(Y)}} = C' \cap f^{-1}(Y)$. Thus $a \in C'$ and $a \in f^{-1}(Y)$. Hence $a \notin (C')^c$. Since f is an injective, then $f(a) \notin f((C')^c) = (f(C'))^c$. Since (Y, τ_2) is a OT_3-space, there exist $U, V \in S(\tau_2)$ such that $f(C')^c \subseteq U, f(a) \in V$ and $U \cap V = \emptyset$. Since f is an ordinary smooth continuous and $(f^{-1}(Y), \tau_{1f^{-1}(Y))})$ is an ordinary smooth subspace of (X, τ_1) and $f^{-1}(U), f^{-1}(V) \subseteq X$, then

$$0 < \tau_2(U) \leq \tau_1(f^{-1}(U)) \leq \tau_{1f^{-1}(Y)}(f^{-1}(U)),$$

$$0 < \tau_1(V) \leq \tau_1(f^{-1}(V)) \leq \tau_{1f^{-1}(Y)}(f^{-1}(V)).$$

Thus $f^{-1}(U), f^{-1}(V) \in S(\tau_{1f^{-1}(Y)})$. Since f is an injective, then $a \in f^{-1}(V)$, $(C')^c \subseteq f^{-1}(U)$ and $f^{-1}(U) \cap f^{-1}(V) = \emptyset$. Hence $(f^{-1}(Y), \tau_{1f^{-1}(Y)})$ is a OT_3-space. □

3.5 OT_4-*spaces*

In this section, we will introduce the notion of OT_4-spaces and investigate some of their properties.

Definition 3.5.1. An osts (X, τ) is called a OT_4-*space* if and only if for each $A, B \subseteq X$ are disjoint in X, satisfying $\tau(A^c) > 0, \tau(B^c) > 0$, there exist $U, V \in S(\tau)$ such that $A \subseteq U$, $B \subseteq V$ and $U \cap V = \emptyset$.

Example 3.5.2. Let $X = \{a, b, c\}$ and we define the mapping $\tau : 2^X \to I$ as follows: $\tau(X) = \tau(\emptyset) = 1, \tau(\{a\}) = 0.6, \tau(\{b\}) = 0.4, \tau(\{a, b\}) = 0.5$ and $\tau(A) = 0$ if $A \notin \{X, \emptyset, \{a\}\{b\}, \{a, b\}\}$. Clearly, (X, τ) is an osts. Let consider $X, \emptyset, \{c\}, \{a, b\}$ and $\{a, c\}$ such that $X, \emptyset, \{c\}^c, \{a, b\}^c \in S(\tau)$. Then $A \cap B \neq \emptyset$ for all $A, B \subseteq X$ such that $\tau(A^c) > 0, \tau(B^c) > 0$ which $A \neq B$ and $A, B \neq \emptyset$. Hence $(X, \tau\ )$ is a OT_4-space. Furthermore, $(X, \tau\ )$ is a not a OT_3-space. Since $a \notin \{c\}$, which $\tau(\{c\}) > 0$ and there exists $X \in S(\tau)$ contains a.

Definition 3.5.3. An osts (X, τ) is said to be a OT-*normal space* if (X, τ) is a OT_4-space and OT_1-space.

Theorem 3.5.4. Let an osts (X, τ) is a OT-normal space, then (X, τ) is a OT-regular space.

Proof. Let (X, τ) be an OT-normal space. Then (X, τ) is a OT_4-space and OT_1-space. We will show that (X, τ) is a OT-regular space. sufficient to proof that (X, τ) is a OT_3-space. Let $F \subseteq X$, satisfying $\tau(F^c) > 0$, and let $x \in X$, satisfying $x \notin F$, then $x \in X \backslash F$. Since $F \cap F^c = \emptyset$, then $\{x\} \cap F = \emptyset$. Since (X, τ) is a OT_1-space, then $\{x\} = \overline{\{x\}}$, then $\overline{\{x\}} \cap F = \emptyset$. Hence, there exists $U \in 2^X$ such that $\{x\} \subseteq U,\ \tau(U^c) > 0$ and $F \cap U = \emptyset$. Since (X, τ) is a OT_4-space, then there exist $W, Z \in S(\tau)$ such that $U \subseteq W,\ F \subseteq Z$ and $W \cap Z = \emptyset$. Since $x \in U$, then $x \in W$. Hence (X, τ) is a OT_3-space. Therefore (X, τ) is a OT-regular space. □

Theorem 3.5.5. Every subspace of OT_4-space is also OT_4-space.

Proof. Let an osts (X, τ) be a OT_4-space and let (A, τ_A) be an ordinary smooth subspace of (X, τ) and let $E, F \subseteq A$, satisfying $E^{c_A}, F^{c_A} \in S(\tau_A)$ and $E \cap F = \emptyset$. Since

$$\tau_A(E^{c_A}) = \bigvee\{\tau(C) : C \in 2^X \text{ and } C \cap A = E^{c_A}\},$$

$$\tau_A(F^{c_A}) = \bigvee\{\tau(C) : C \in 2^X \text{ and } C \cap A = F^{c_A}\},$$

then $C \cap A = E^{c_A}, C \cap A = F^{c_A}$. Hence $A \cap E^{c_A} \subseteq C, A \cap F^{c_A} \subseteq C$ and $E \cap C = \emptyset, F \cap C = \emptyset$. Let $\tau_A(E^{c_A}) = \delta_1$ and $\tau_A(F^{c_A}) = \delta_2$. Since

$$\delta_1 = \bigvee\{\tau(C) : C \in 2^X \text{ and } C \cap A = E^{c_A}\},$$

$$\delta_2 = \bigvee\{\tau(C) : C \in 2^X \text{ and } C \cap A = F^{c_A}\},$$

then there exist $C', C'' \subseteq 2^X, C' \cap A = E^{c_A}$ and $C'' \cap A = F^{c_A}$ such that

$$\begin{aligned} \tau(C') &> \tau_A(E^{c_A}) - \frac{\delta_1}{2} \\ &= \delta_1 - \frac{\delta_1}{2} \\ &= \frac{\delta_1}{2} \\ &> 0, \end{aligned}$$

$$\begin{aligned}\tau(C'') &> \tau_A(F^{c_A}) - \frac{\delta_2}{2}\\ &= \delta_2 - \frac{\delta_2}{2}\\ &= \frac{\delta_1}{2}\\ &> 0.\end{aligned}$$

Since (X, τ) is a OT_4-space, there exist $U, V \in S(\tau)$ such that $(C')^c \subseteq U, (C'')^c \subseteq V$ and $U \cap V = \emptyset$. Let $W = U \cap A$ and $Z = V \cap A$,

$$\begin{aligned}\tau_A(W) &= \bigvee\{\tau(U) : U \in 2^X \text{ and } U \cap A = W\}\\ &\geq \tau(U)\\ &> 0,\end{aligned}$$

$$\begin{aligned}\tau_A(Z) &= \bigvee\{\tau(V) : V \in 2^X \text{ and } V \cap A = Z\}\\ &\geq \tau(V)\\ &> 0,\end{aligned}$$

and

$$\begin{aligned}W \cap Z &= (U \cap A) \cap (V \cap A),\\ &= (U \cap V) \cap A,\\ &= \emptyset \cap A,\\ &= \emptyset.\end{aligned}$$

So $W, Z \in S(\tau_A)$ such that $E \subseteq W, F \subseteq Z$ and $W \cap Z = \emptyset$. Hence (A, τ_A) is a OT_4-space. □

The next theorem, we give the equivalent conditions to be OT_4-spaces.

Theorem 3.5.6. Let (X, τ) be an osts. Then the following conditions are equivalent:

(i) (X, τ) is a OT_4-space.

(ii) If $U \subseteq X$, satisfying $\tau(U) > 0$ is a superset of a set A, satisfying $\tau(A^c) > 0$, then there exists V, satisfying $\tau(V) > 0$ such that $A \subseteq V \subseteq \overline{V} \subseteq U$.

(iii) For each pair of disjoint sets A, B satisfying $\tau(A^c) > 0, \tau(B^c) > 0$, there exists U, satisfying $\tau(U) > 0$ which $A \subseteq U$ and $\overline{U} \cap B = \emptyset$.

(iv) For each pair of disjoint sets $A, B \subseteq X$, satisfying $\tau(A^c) > 0, \tau(B^c) > 0$, there exist sets $U, V \in S(\tau)$ such that $A \subseteq U, B \subseteq V$ and $\overline{U} \cap \overline{V} = \emptyset$.

Proof. (i) $\Rightarrow$ (ii) Let $A \subseteq X$, satisfying $\tau(A^c) > 0$ and $U \subseteq X$, satisfying $\tau(U) > 0$ such that $A \subseteq U$. Let $F = X\backslash U$ which

$$\tau((X\backslash F)) = \tau(X\backslash(X\backslash U)) = \tau(U) > 0,$$

and $A \cap F = \emptyset$. Since (X, τ) is a OT_4-space, there exist $V, W \in S(\tau)$ such that $A \subseteq V, F \subseteq W$ and $V \cap W = \emptyset$. So $V \subseteq X\backslash W$. By Proposition 2.3.7 (i), then $\overline{V} \subseteq \overline{X\backslash W}$. And by Proposition 2.3.10 (ii), then $\overline{X\backslash W} = X\backslash W$. Thus $\overline{V} \subseteq X\backslash W$, but $X\backslash W \subseteq X\backslash F = U$. Hence $\overline{V} \subseteq U$. Therefore $A \subseteq V \subseteq \overline{V} \subseteq U$.

(ii) $\Rightarrow$ (iii) Let $A, B \subseteq X$, satisfying $\tau(A^c) > 0, \tau(B^c) > 0$ which $A \cap B = \emptyset$. By (ii), then $A \subseteq U \subseteq \overline{U} \subseteq X\backslash B$. Hence $\overline{U} \cap B = \emptyset$.

(iii) $\Rightarrow$ (iv) Let $A, B \subseteq X$, satisfying $\tau(A^c) > 0, \tau(B^c) > 0$ which $A \cap B = \emptyset$. By (iii), then there exists $U, \tau(U) > 0$ with $A \subseteq U$ and $\overline{U} \cap B = \emptyset$. Since $\overline{U} = \cap\{F \in 2^X : U \subseteq F \text{and } \tau(F^c) > 0\}$, then there exists $F \in 2^X$ such that $U \subseteq F,\ \tau(F^c) > 0$ and $B \cap F = \emptyset$. By assumption, then there exists $V \in S(\tau)$, $B \subseteq V$ and $\overline{V} \cap F = \emptyset$. Consider,

$$\begin{aligned} \overline{V} \cap \overline{U} &\subseteq \overline{V} \cap \overline{F} \\ &= \overline{V} \cap F \\ &= \emptyset. \end{aligned}$$

Hence $\overline{V} \cap \overline{U} = \emptyset$. Therefore Hence (iv) is true.

(iv) $\Rightarrow$ (i) Let $A, B \subseteq X$, satisfying $\tau(A^c) > 0, \tau(B^c) > 0$ and $A \cap B = \emptyset$. By (iv), then there exist set $U, V \in S(\tau)$ such that $A \subseteq U, B \subseteq V$ and $\overline{U} \cap \overline{V} = \emptyset$. Since $U \subseteq \overline{U}$ and $V \subseteq \overline{V}$ then $U \cap V = \emptyset$. Hence (X, τ) is a OT_4-space. $\square$

The following results are the properties of OT_4-spaces under some kinds of ordinary smooth maps.

Proposition 3.5.7. Let (X, τ_1) and (Y, τ_2) be two osts and let $f : X \to Y$ be an ordinary smooth homeomorphism. Then (X, τ_2) is an OT_4-space if and only if (Y, τ_2) is an OT_4-space.

Proof. ($\Longrightarrow$) : Let A, B are disjoint in Y, satisfying $A^c, B^c \in S(\tau_2)$. Since f is a bijective and an ordinary smooth continuous, then

$$f^{-1}(A) \cap f^{-1}(B) = \emptyset$$

$$\tau_1(f^{-1}(A)^c) = \tau_1(f^{-1}(A^c)) \geq \tau_2(A^c) > 0,$$

and

$$\tau_1(f^{-1}(B)^c) = \tau_1(f^{-1}(B^c)) \geq \tau_2(B^c) > 0.$$

So $f^{-1}(A)^c, f^{-1}(B)^c \in S(\tau_1)$. Since (X, τ_1) is a OT_4-space, then there exist $U, V \in S(\tau_1)$ such that $f^{-1}(A) \subseteq U, f^{-1}(B) \subseteq V$ and $U \cap V = \emptyset$. Since f is an ordinary smooth open, it follows that

$$0 < \tau_1(U) \leq \tau_2(f(U))$$

and

$$0 < \tau_1(V) \leq \tau_2(f(V)).$$

Thus $f(U), f(V) \in S(\tau_2)$. Since f is a bijective, then $A \subseteq f(U),\ B \subseteq f(V)$ and $f(U) \cap f(V) = \emptyset$. Hence (Y, τ_2) is a OT_4-space.

($\Longleftarrow$) : Let A, B are disjoint in X, satisfying $A^c, B^c \in S(\tau_1)$. Since f is a bijective and an ordinary smooth closed, then

$$f(A) \cap f(B) = \emptyset$$

$$\tau_2(f(A)^c) = \tau_2(f(A^c)) \geq \tau_1(A^c) > 0,$$

and

$$\tau_2(f(B)^c) = \tau_2(f(B^c)) \geq \tau_1(B^c) > 0.$$

So $f(A)^c, f(B)^c \in S(\tau_2)$. Since (Y, τ_2) is a OT_4-space, then there exist $U, V \in S(\tau_2)$ such that $f(A) \subseteq U, f(B) \subseteq V$ and $U \cap V = \emptyset$. Since f is an ordinary smooth continuous, it follows that

$$0 < \tau_2(U) \leq \tau_1(f^{-1}(U))$$

and

$$0 < \tau_2(V) \leq \tau_1(f^{-1}(V)).$$

Thus $f^{-1}(U), f^{-1}(V) \in S(\tau_1)$. Since f is a bijective, then $A \subseteq f^{-1}(U),\ B \subseteq f^{-1}(V)$ and $f^{-1}(U) \cap f^{-1}(V) = \emptyset$. Hence (X, τ_1) is a OT_4-space. □

Proposition 3.5.8. Let $f : X \to Y$ be an injective, ordinary smooth closed and ordinary smooth continuous map with respect to the ordinary smooth topologies τ_1 and τ_2 respectively. If (Y, τ_2) is a OT_4-space, then so is a (X, τ_1).

Proof. Let $A, B \subseteq X$, satisfying $A^c, B^c \in S(\tau_1)$. Since f is an injective and ordinary smooth closed, it follows that

$$f(A) \cap f(B) = \emptyset,$$

$$\tau_2(f(A)^c) \geq \tau_1(A)^c > 0,$$

$$\tau_2(f(B)^c) \geq \tau_1(B)^c > 0.$$

Since (Y, τ_2) is a OT_4-space, there exist $U, V \in S(\tau_2)$ such that $f(A) \subseteq U, f(B) \subseteq V$ and $U \cap V = \emptyset$. Since f is an injective and an ordinary smooth continuous, it follows that

$$\tau_1(f^{-1}(U)) \geq \tau_2(U) > 0,$$

$$\tau_1(f^{-1}(V)) \geq \tau_2(V) > 0,$$

$$f^{-1}(U) \cap f^{-1}(V) = \emptyset,$$

$$A = f^{-1}(f(A)) \subseteq f^{-1}(U)$$

and

$$B = f^{-1}(f(B)) \subseteq f^{-1}(V).$$

So $f^{-1}(U), f^{-1}(V) \in S(\tau_1)$ such that $A \subseteq f^{-1}(U), B \subseteq f^{-1}(V)$ and $f^{-1}(U) \cap f^{-1}(V) = \emptyset$. Hence (X, τ_1) is a OT_4-space. □

Proposition 3.5.9. An osts (X, τ_1) is a OT_4-space. If $f : X \to Y$ is an injective, ordinary smooth continuous and ordinary smooth open, then $(f(X), \tau_{2f(X)})$ is a OT_4-space.

Proof. Let A, B are disjoint in $f(X)$, satisfying $A^{c_{f(X)}}, B^{c_{f(X)}} \in S(\tau_{2f(X)})$. Since f is an injective, there exist $f^{-1}(A), f^{-1}(B)$ are disjoint in X. Since

$$\tau_{2f(X)}(A^{c_f(X)}) = \bigvee\{\tau_2(C) : C \in 2^Y \text{ and } C \cap f(X) = A^{c_{f(X)}}\},$$

$$\tau_{2f(X)}(A^{c_f(X)}) = \bigvee\{\tau_2(C) : C \in 2^Y \text{ and } C \cap f(X) = B^{c_f(X)}\},$$

then $C \cap f(X) = A^{c_{f(X)}}, C \cap f(X) = B^{c_{f(X)}}$. Hence $f(X) \cap A^{c_{f(X)}} \subseteq C, f(X) \cap B^{c_{f(X)}} \subseteq C$ and $A \cap C = \emptyset, B \cap C = \emptyset$. Let $\tau_{2f(X)}(A^{c_{f(X)}}) = \delta_1$ and $\tau_{2f(X)}(B^{c_{f(X)}}) = \delta_2$. Then

$$\delta_1 = \bigvee\{\tau_2(C) : C \in 2^Y \text{ and } C \cap f(X) = A^{c_{f(X)}}\},$$

$$\delta_2 = \bigvee\{\tau_2(C) : C \in 2^Y \text{ and } C \cap f(X) = B^{c_{f(X)}}\}.$$

Thus, there exist $C', C'' \subseteq 2^Y, C' \cap f(X) = A^{c_{f(X)}}, C'' \cap f(X) = B^{c_{f(X)}}$ such that

$$\begin{aligned}\tau_2(C') &> \tau_{2f(X)}(A^{c_{f(X)}}) - \frac{\delta_1}{2}\\ &= \delta_1 - \frac{\delta_1}{2}\\ &= \frac{\delta_1}{2}\\ &> 0\end{aligned}$$

$$\begin{aligned}\tau_2(C'') &> \tau_{2f(X)}(A^{c_{f(X)}}) - \frac{\delta_2}{2}\\ &= \delta_2 - \frac{\delta_2}{2}\\ &= \frac{\delta_2}{2}\\ &> 0.\end{aligned}$$

Since f is an ordinary smooth continuous we have

$$\tau_1(f^{-1}(C')) \geq \tau_2((C')) > 0,$$

$$\tau_1(f^{-1}(C'')) \geq \tau_2((C'')) > 0.$$

Since (X, τ_1) is a OT_4-space, there exist $U, V \in S(\tau_1)$ such that $f^{-1}(C')^c \subseteq U, f^{-1}(C'')^c \subseteq V$ and $U \cap V = \emptyset$. Since f is an ordinary smooth open and $(f(X), \tau_{2f(X)})$ is an ordinary smooth subspace of (Y, τ_2) and $f(U), f(V) \subseteq f(X)$, then

$$0 < \tau_1(U) \leq \tau_2(f(U)) \leq \tau_{2f(X)}(f(U)),$$

$$0 < \tau_1(V) \leq \tau_2(f(V)) \leq \tau_{2f(X)}(f(V)).$$

Thus $f(U), (f(V)) \in S(\tau_{2f(X)})$. Since f is an injective, then $A \subseteq (C')^c \subseteq f(U), B \subseteq (C'')^c \subseteq f(V)$ and $f(U) \cap f(V) = \emptyset$. Hence, $(f(X), \tau_{2f(X)})$ is a OT_4-space. □

Proposition 3.5.10. An osts (Y, τ_2) is a OT_4-space. If $f : X \to Y$ is an injective, ordinary smooth open and ordinary smooth continuous, then $(f^{-1}(Y), \tau_{2f^{-1}(Y)})$ is a OT_4-space.

Proof. Let A, B are disjoint in $f^{-1}(Y)$, satisfying $A^{c_{f^{-1}(Y)}}, B^{c_{f^{-1}(Y)}} \in S(\tau_{1f^{-1}(Y)})$. Since f is an injective, there exist $f(A), f(B)$ are disjoint in Y. Since

$$\tau_{1f^{-1}(Y)}(A^{c_f^{-1}(Y)}) = \bigvee\{\tau_1(C) : C \in 2^X \text{ and } C \cap f^{-1}(Y) = A^{c_{f^{-1}(Y)}}\},$$

$$\tau_{1f^{-1}(Y)}(A^{c_f^{-1}(Y)}) = \bigvee\{\tau_1(C) : C \in 2^X \text{ and } C \cap f^{-1}(Y) = B^{c_f^{-1}(Y)}\},$$

then $C \cap f^{-1}(Y) = A^{c_{f^{-1}(Y)}}, C \cap f^{-1}(Y) = B^{c_{f^{-1}(Y)}}$. Hence $f^{-1}(Y) \cap A^{c_{f^{-1}(Y)}} \subseteq C, f^{-1}(Y) \cap B^{c_{f^{-1}(Y)}} \subseteq C$ and $A \cap C = \emptyset, B \cap C = \emptyset$. Let $\tau_{1f^{-1}(Y)}(A^{c_{f^{-1}(Y)}}) = \delta_1$ and $\tau_{1f^{-1}(Y)}(B^{c_{f^{-1}(Y)}}) = \delta_2$. Then

$$\delta_1 = \bigvee\{\tau_1(C) : C \in 2^X \text{ and } C \cap f^{-1}(Y) = A^{c_{f^{-1}(Y)}}\},$$

$$\delta_2 = \bigvee\{\tau_1(C) : C \in 2^X \text{ and } C \cap f^{-1}(Y) = B^{c_{f^{-1}(Y)}}\}.$$

Thus, there exists $C', C'' \subseteq 2^X, C' \cap f^{-1}(Y) = A^{c_{f^{-1}(Y)}}, C'' \cap f^{-1}(Y) = B^{c_{f^{-1}(Y)}}$ such that

$$\begin{aligned}\tau_1(C') &> \tau_{1f^{-1}(Y)}(A^{c_{f^{-1}(Y)}}) - \frac{\delta_1}{2} \\ &= \delta_1 - \frac{\delta_1}{2} \\ &= \frac{\delta_1}{2} \\ &> 0\end{aligned}$$

$$\begin{aligned}\tau_1(C'') &> \tau_{1f^{-1}(Y)}(A^{c_{f^{-1}(Y)}}) - \frac{\delta_2}{2} \\ &= \delta_2 - \frac{\delta_2}{2}\end{aligned}$$

$$= \frac{\delta_2}{2}$$
$$> 0.$$

Since f is an ordinary smooth open we have

$$\tau_2(f(C')) \geq \tau_1(C') > 0,\ \tau_2(f(C'')) \geq \tau_1(C'') > 0$$

Since (Y, τ_2) is a OT_4-space, there exist $U, V \in S(\tau_2)$ such that $f(C')^c \subseteq U, f(C'')^c \subseteq V$ and $U \cap V = \emptyset$. Since f is an ordinary smooth continuous and $(f^{-1}(Y), \tau_{1f^{-1}(Y)})$ is an ordinary smooth subspace of (X, τ_1) and $f^{-1}(U), f^{-1}(V) \subseteq X$, then

$$0 < \tau_2(U) \leq \tau_1(f^{-1}(U)) \leq \tau_{1f^{-1}(Y)}(f^{-1}(U)),$$

$$0 < \tau_2(V) \leq \tau_1(f^{-1}(V)) \leq \tau_{2f^{-1}(Y)}(f^{-1}(V)).$$

Thus $f^{-1}(U), (f^{-1}(V)) \in S(\tau_{1f^{-1}(Y)})$. Since f is an injective, then $A \subseteq (C')^c \subseteq f^{-1}(U), B \subseteq (C'')^c \subseteq f^{-1}(V)$ and $f^{-1}(U) \cap f^{-1}(V) = \emptyset$. Hence, $(f^{-1}(Y), \tau_{1f^{-1}(Y)})$ is a OT_4-space. □

3.6 OT_5-*spaces*

In this section, we will introduce the notion of OT_5-spaces and investigate some of their properties.

Definition 3.6.1. An osts (X, τ) is called a OT_5-*space* if and only if for each A, B are separated sets in X ($\overline{A} \cap B = \emptyset$ and $A \cap \overline{B} = \emptyset$) there exist $U, V \in S(\tau)$ such that $A \subseteq U$, $B \subseteq V$ and $U \cap V = \emptyset$.

Example 3.6.2. Let $X = \{a, b, c\}$ and we define the mapping $\tau : 2^X \to I$ as follows: $\tau(X) = \tau(\emptyset) = 1, \tau(\{b\}) = 0.6, \tau(\{a, c\}) = 0.4$ and $\tau(A) = 0$ if $A \notin \{X, \emptyset, \{b\}, \{a, c\}\}$. Clearly, (X, τ) is an osts. Since $\overline{\{b\}} = \{b\}$ and $\overline{\{a, c\}} = \{a, c\}$, then $\overline{\{b\}} \cap \{a, c\} = \emptyset$ and $\{b\} \cap \overline{\{a, c\}} = \emptyset$. Hence $\{b\}, \{a, c\}$ are separated sets in X and $\{b\}, \{a, c\} \in S(\tau)$. Therefore (X, τ) is a OT_5-space.

Theorem 3.6.3. Let an osts (X, τ) be a OT_5-space, then (X, τ) is a OT_4-space.

Proof. Let (X, τ) be an OT_5-space and $A, B \subseteq X$, with $A \cap B = \emptyset$ such that $\tau(A^c) > 0, \tau(B^c) > 0$. By Proposition 2.3.10 (ii), then $\overline{A} = A, \overline{B} = B$. Since $A \cap B = \emptyset$ we have

$\overline{A} \cap B = \emptyset$ and $A \cap \overline{B} = \emptyset$. Since (X, τ) is a OT_5-space, there exist $U, V \in S(\tau)$ such that $A \subseteq U$, $B \subseteq V$ and $U \cap V = \emptyset$. Thus (X, τ) is a OT_4-space. □

Theorem 3.6.4. Every subspace of OT_5-space is also OT_5-space.

Proof. Let an osts (X, τ) is a OT_5-space and let (A, τ_A) be an ordinary smooth subspace of (X, τ) and let M, N are separated sets in A, ($\overline{M} \cap N = \emptyset$ and $M \cap \overline{N} = \emptyset$). Since $A \subseteq X$, then $M, N \subseteq X$. Since $\overline{M}, \overline{N} in X$ are subset of $\overline{M}, \overline{N} \in A$, respectively. Then M, N are separated sets in X. And since (X, τ) is a OT_5-space, there exist $U, V \in S(\tau)$ such that $M \subseteq U, N \subseteq V$ and $U \cap V = \emptyset$. Let $W = U \cap A$ and $Z = V \cap A$,

$$\begin{aligned}\tau_A(W) &= \bigvee\{\tau(U) : U \in 2^Y \text{ and } U \cap A = W\}\\ &\geq \tau(U)\\ &> 0\end{aligned}$$

$$\begin{aligned}\tau_A(Z) &= \bigvee\{\tau(V) : V \in 2^Y \text{ and } V \cap A = Z\}\\ &\geq \tau(V)\\ &> 0\end{aligned}$$

and

$$\begin{aligned}W \cap Z &= (U \cap A) \cap (V \cap A),\\ &= (U \cap V) \cap A,\\ &= \emptyset \cap A,\\ &= \emptyset.\end{aligned}$$

So $W, Z \in S(\tau_A)$ such that $M \subseteq W, N \subseteq Z$ and $W \cap Z = \emptyset$. Hence (A, τ_A) is a OT_5-space. □

The following results are the properties of OT_5-spaces under some kinds of ordinary smooth maps.

Proposition 3.6.5. Let (X, τ_1) and (Y, τ_2) be two osts and let $f : X \to Y$ be an ordinary smooth homeomorphism. Then (X, τ_2) is an OT_5-space if and only if $(Y, \tau_2)$is an OT_5-space.

Proof. ($\Longrightarrow$) : Let A, B are separated sets in Y. Since f is a bijective,ordinary smooth continuous and ordinary smooth closed, then

$$\overline{f^{-1}(A)} \cap f^{-1}(B) = f^{-1}(\overline{A}) \cap f^{-1}(B) = \overline{A} \cap B = \emptyset$$

and

$$f^{-1}(A) \cap \overline{f^{-1}(B)} = f^{-1}(A) \cap f^{-1}(\overline{B}) = A \cap \overline{B} = \emptyset.$$

So $f^{-1}(A), f^{-1}(B)$ are separated sets in X. Since (X, τ_1) is a OT_5-space, then there exist $U, V \in S(\tau_1)$ such that $f^{-1}(A) \subseteq U,\ f^{-1}(B) \subseteq V$ and $U \cap V = \emptyset$. Since f is ordinary smooth open, it follows that

$$0 < \tau_1(U) \leq \tau_2(f(U))$$

and

$$0 < \tau_1(V) \leq \tau_2(f(V)).$$

Thus $f(U), f(V) \in S(\tau_2)$. Since f is a injective, then

$$A = f(f^{-1}(A)) \subseteq f(U),$$

$$B = f(f^{-1}(B)) \subseteq f(V)$$

and

$$f(U) \cap f(V) = \emptyset.$$

Hence (Y, τ_2) is a OT_5-space.

($\Longleftarrow$) : Let A, B are separated sets in X. Since f is a bijective,ordinary smooth continuous and ordinary smooth closed, then

$$\overline{f(A)} \cap f(B) = f(\overline{A}) \cap f(B) = \overline{A} \cap B = \emptyset$$

and

$$f(A) \cap \overline{f(B)} = f(A) \cap f(\overline{B}) = A \cap \overline{B} = \emptyset.$$

So $f(A), f(B)$ are separated sets in Y. Since (Y, τ_2) is a OT_5-space, then there exist $U, V \in S(\tau_2)$ such that $f(A) \subseteq U,\ f(B) \subseteq V$ and $U \cap V = \emptyset$. Since f is ordinary smooth continuous, it follows that

$$0 < \tau_2(U) \leq \tau_1(f^{-1}(U))$$

and

$$0 < \tau_2(V) \leq \tau_1(f^{-1}(V)).$$

Thus $f^{-1}(U), f^{-1}(V) \in S(\tau_1)$. Since f is a injective, then

$$A = f^{-1}(f(A)) \subseteq f^{-1}(U),$$

$$B = f^{-1}(f(B)) \subseteq f^{-1}(V)$$

and

$$f^{-1}(U) \cap f^{-1}(V) = \emptyset.$$

Hence (X, τ_1) is a OT_5-space. □

Proposition 3.6.6. Let $f : X \to Y$ be an injective,ordinary smooth closed and ordinary smooth continuous map with respect to the ordinary smooth topologies τ_1 and τ_2 respectively. If (Y, τ_2) is a OT_5-space, then so is a (X, τ_1).

Proof. Let A, B are separated set in X. Since f is a bijective,ordinary smooth continuous and ordinary smooth closed, then

$$\overline{f(A)} \cap f(B) = f(\overline{A}) \cap f(B) = \overline{A} \cap B = \emptyset$$

and

$$f(A) \cap \overline{f(B)} = f(A) \cap f(\overline{B}) = A \cap \overline{B} = \emptyset.$$

So $f(A), f(B)$ are separated sets in Y. Since (Y, τ_2) is a OT_5-space, then there exist $U, V \in S(\tau_2)$ such that $f(A) \subseteq U, f(B) \in V$ and $U \cap V = \emptyset$. Since f is an injective and ordinary smooth continuous, it follows that

$$\tau_1(f^{-1}(U)) \geq \tau_2(U) > 0,$$

$$\tau_1(f^{-1}(V)) \geq \tau_2(V) > 0,$$

$$f^{-1}(U) \cap f^{-1}(V) = \emptyset,$$

$$A = f^{-1}(f(A)) \subseteq f^{-1}(U)$$

and

$$B = f^{-1}(f(B)) \subseteq f^{-1}(V).$$

Hence (X, τ_1) is a OT_5-space. □

Proposition 3.6.7. An osts (X, τ_1) is a OT_5-space. If $f : X \to Y$ is an injective, ordinary smooth continuous, ordinary smooth closed and ordinary smooth open, then $(f(X), \tau_{2f(X)})$ is a OT_5-space.

Proof. Let A, B are separated set in $f(X)$. Since $\overline{A}, \overline{B}$ in Y are subset of $\overline{A}, \overline{B}$ in $f(X)$ respectively, then A, B are separated set in Y. Since f is an injective, ordinary smooth continuous and ordinary smooth closed, then

$$\overline{f^{-1}(A)} \cap f^{-1}(B) = f^{-1}(\overline{A}) \cap f^{-1}(B) = \overline{A} \cap B = \emptyset$$

and

$$f^{-1}(A) \cap \overline{f^{-1}(B)} = f^{-1}(A) \cap f^{-1}(\overline{B}) = A \cap \overline{B} = \emptyset.$$

Hence $f^{-1}(A), f^{-1}(B)$ are separated set in X. Since (X, τ_1) is a OT_5-space, then there exist $U, V \in S(\tau_1)$ such that $f^{-1}(A) \subseteq U, f^{-1}(B) \subseteq V$ and $U \cap V = \emptyset$. Since f is an ordinary smooth open and $(f(X), \tau_{2f(X)})$ is an ordinary smooth subspace of (Y, τ_2) and $f(U), f(V) \subseteq f(X)$, then

$$0 < \tau_1(U) \leq \tau_2(f(U)) \leq \tau_{2f(X)}(f(U)),$$

$$0 < \tau_1(V) \leq \tau_2(f(V)) \leq \tau_{2f(X)}(f(V)).$$

Since f is an injective, then

$$A \subseteq f(U), B \subseteq f(V),$$

$$f(U) \cap f(V) = \emptyset.$$

So, there exist $f(U), (f(V)) \in S(\tau_{2f(X)})$ such that $A \subseteq f(U), B \subseteq f(V)$ and $f(U) \cap f(V) = \emptyset$. Hence, $(f(X), \tau_{f(X)})$ is a OT_5-space. □

Proposition 3.6.8. An osts (Y, τ_2) is a OT_5-space. If $f : X \to Y$ is injective, ordinary smooth continuous and ordinary smooth open, then $(f^{-1}(Y), \tau_{1f^{-1}(Y)})$ is a OT_5-space.

Proof. Let A, B are separated sets in $f^{-1}(Y)$. Since $\overline{A}, \overline{B}$ in X are subset of $\overline{A}, \overline{B}$ in $f^{-1}(Y)$ respectively, then A, B are separated sets in X. Since f is an injective, ordinary smooth continuous and ordinary smooth closed, then

$$\overline{f(A)} \cap f(B) = f(\overline{A}) \cap f(B) = \overline{A} \cap B = \emptyset$$

and

$$f(A) \cap \overline{f(B)} = f(A) \cap f(\overline{B}) = A \cap \overline{B} = \emptyset.$$

Hence $f(A), f(B)$ are separated set in Y. Since (Y, τ_2) is a OT_5-space, then there exist $U, V \in S(\tau_2)$ such that $f(A) \subseteq U, f(B) \subseteq V$ and $U \cap V = \emptyset$. Since f is an ordinary smooth continuous and $(f^{-1}(Y), \tau_{1f^{-1}(Y)})$ is an ordinary smooth subspace of (X, τ_1) and $f^{-1}(U), f^{-1}(V) \subseteq X$, then

$$0 < \tau_2(U) \leq \tau_1(f^{-1}(U)) \leq \tau_{1f^{-1}(Y)}(f^{-1}(U)),$$

$$0 < \tau_2(V) \leq \tau_1(f^{-1}(V)) \leq \tau_{1f^{-1}(Y)}(f^{-1}(V)).$$

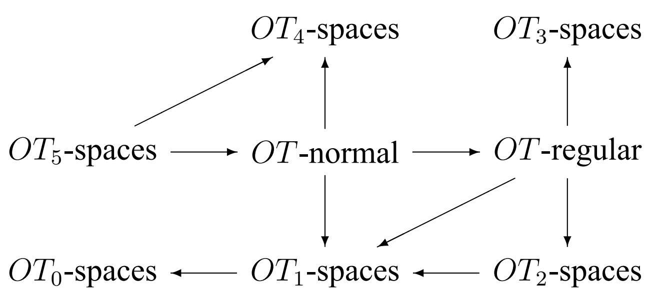
Since f is an injective

$$A \subseteq f^{-1}(U), B \subseteq f^{-1}(V),$$

$$f^{-1}(U) \cap f^{-1}(V) = \emptyset.$$

So, there exist $f^{-1}(U), f^{-1}(V) \in S(\tau_{1f^{-1}(Y)})$ such that $A \subseteq f^{-1}(U), B \subseteq f^{-1}(V)$ and $f^{-1}(U) \cap f^{-1}(V) = \emptyset$. Hence, $(f^{-1}(Y), \tau_{1f^{-1}(Y)})$ is a OT_5-space. □

The following diagram illustrates the relationship between the spaces discussed in this chapter.



CHAPTER 4

OST-DENSE SETS

In this chapter, we introduce the concepts of OST-dense sets on ordinary smooth topological spaces and study some fundamental of their properties.

4.1 OST-*dense sets*

In this section, we will introduce the notion of OST-dense sets on ordinary smooth topological spaces and investigate some of their properties.

Definition 4.1.1. A set A is a OST-*dense set* in X if and only if $\overline{A} = X$.

Example 4.1.2. Let $X = \{1, 2, 3\}$ and we define the mapping $\tau : 2^X \to I$ as follows: $\tau(X) = \tau(\emptyset) = 1, \tau(\{1\}) = 0.7, \tau(\{3\}) = 0.4, \tau(\{1, 3\}) = 0.5, \tau(\{2, 3\}) = 0.3$ and $\tau(A) = 0$ if $A \notin \{X, \emptyset, \{1\}, \{3\}, \{1, 3\}, \{2, 3\}\}$. Thus $\overline{\{1, 3\}} = X$, and we have $\{1, 3\}$ is a OST-dense set in X.

The following results therefore follows directly from the definition of OST-dense sets.

Lemma 4.1.3. A set A is a OST-dense set in X if and only if $U^\circ \cap A \neq \emptyset$ for all subset U of X with $U^\circ \neq \emptyset$.

Proof. ($\Longrightarrow$) : Assume that A is a OST-dense set in X. Then $\overline{A} = X$. We will show that $U^\circ \cap A \neq \emptyset$ for all subset U of X with $U^\circ \neq \emptyset$. Let $U^\circ \neq \emptyset$. Suppose that $U^\circ \cap A = \emptyset$. Since $U^\circ \neq \emptyset$, there exists $V_0 \subseteq U, \tau(V_0) > 0$ and $V_0 \neq \emptyset$. Since $U^\circ \cap A = \emptyset$, then $V_0 \cap A = \emptyset$. So $A \subseteq (V_0)^c$ and $\tau(V_0) > 0$. By Proposition 2.3.7 (i), then $\overline{A} \subseteq \overline{(V_0)^c}$. From Proposition 2.3.10 (ii), we have $\overline{(V_0)^c} = (V_0)^c$. Hence $\overline{A} \subseteq (V_0)^c$. Thus

$$\emptyset \neq V_0 = X \setminus (V_0)^c \subseteq X \setminus \overline{A}.$$

Which contradicts with $\overline{A} = X$. Therefore $U^\circ \cap A \neq \emptyset$.

$(\Longleftarrow)$: Assume that $U^\circ \cap A \neq \emptyset$ for all subset U of X with $U^\circ \neq \emptyset$. To show that $\overline{A} = X$. Since $U^\circ \cap A \neq \emptyset$, then $A \neq \emptyset$. So $\overline{A} \neq \emptyset$. Suppose that $\overline{A} \neq X$. Then $X \setminus \overline{A} \neq \emptyset$. Since $X \setminus \overline{A} = (X \setminus A)^\circ$, by assumption,

$$\emptyset \neq (X \setminus A)^\circ \cap A \subseteq (X \setminus A) \cap A.$$

It is a contradiction. Therefore $\overline{A} = X$. □

Theorem 4.1.4. Let (X, τ) is an osts and $A \subseteq X$. Then the following conditions are equivalent:

(i) A is a OST-dense set in X.

(ii) If F is a nonempty subset of X, satisfying $\tau(F^c) > 0$ and $A \subseteq F$, then $F = X$.

(iii) If for all subset U of X with $U^\circ \neq \emptyset$, then $U^\circ \cap A \neq \emptyset$.

Proof. (i) $\Rightarrow$ (ii) : Assume that A is a OST-dense set in X.Let F be a nonempty subset of X, satisfying $\tau(F^c) > 0$ and $A \subseteq F$. We will show that $F = X$. By Proposition 2.3.7 (i), we have $\overline{A} \subseteq \overline{F}$. From Proposition 2.3.10 (ii), we have $\overline{F} = F$. Since A is OST-dense set in X, then $\overline{A} = X$. Thus $X = \overline{A} \subseteq \overline{F} = F$ and $F \subseteq X$. Hence $F = X$.
(ii) $\Rightarrow$ (iii) : Assume that (ii) holds. We will show that $U^\circ \cap A \neq \emptyset$ for all subset U of X with $U^\circ \neq \emptyset$. Let U be a subset of X such that $U^\circ \neq \emptyset$. Suppose that $U^\circ \cap A = \emptyset$. Since $U^\circ \neq \emptyset$, there exists $V_0 \subseteq U, \tau(V_0) > 0$ and $V_0 \neq \emptyset$. Since $U^\circ \cap A = \emptyset$, then $V_0 \cap A = \emptyset$. So $A \subseteq (V_0)^c$ and $\tau(V_0) > 0$. By assumption, $(V_0)^c = X$, then $V_0 = \emptyset$. It is a contradiction. Therefore $U^\circ \cap A \neq \emptyset$.
(iii) $\Rightarrow$ (i) : It follows from Lemma 4.1.3. □

Theorem 4.1.5. Let X be an osts and A be a subset of X. Then A is OST-dense set in X if and only if $(X \setminus A)^\circ = \emptyset$.

Proof. $(\Longrightarrow)$: Assume that A is OST-dense set in X, i.e., $\overline{A} = X$. Then $X \setminus \overline{A} = \emptyset$. By Proposition 2.3.7 (v), then $X \setminus \overline{A} = (X \setminus A)^\circ$. Hence $(X \setminus A)^\circ = \emptyset$.
$(\Longleftarrow)$: Assume that $(X \setminus A)^\circ = \emptyset$. By Proposition 2.3.7 (v), we have $X \setminus \overline{A} = \emptyset$. Hence $\overline{A} = X$. Therefore A is OST-dense set in X. □

CHAPTER 5

CONCLUSIONS

The aim of this thesis is to introduce of concepts of some separation axioms in ordinary smooth topological space by using $S(\tau)$. And we study some properties of $OT_0, OT_1, OT_2, OT_3, OT_4$ and OT_5 spaces. Moreover, we introduce the concepts of OST-dense set on ordinary smooth topological spaces and study some of their properties. The results are follows:

1) An osts (X, τ) is called a OT_0-*space* if and only if for each $x, y \in X$ with $x \neq y$ there exists $U \in S(\tau)$ such that $x \in U$ and $y \notin U$ or $y \in U$ and $x \notin U$. From the above definitions. I have the following theorems are derived:

   1.1) An osts (X, τ) is a OT_0-space if and only if for every $x, y \in X$ such that $x \neq y$ we have that $\overline{\{x\}} \neq \overline{\{y\}}$.

   1.2) Every subspace of OT_0-space is also OT_0-space.

   1.3) Let (X, τ_1) and (Y, τ_2) be two osts and let $f : X \to Y$ be an ordinary smooth homeomorphism. Then (X, τ_1) is an OT_0-space if and only if (Y, τ_2) is an OT_0-space.

   1.4) Let $f : X \to Y$ be an injective, ordinary smooth continuous map with respect to the ordinary smooth topologies τ_1 and τ_2 respectively. If (Y, τ_2) is a OT_0-space, then so is (X, τ_1).

   1.5) An osts (X, τ_1) is a OT_0-space. If $f : X \to Y$ is an injective and ordinary smooth open, then $(f(X), \tau_{2f(X)})$ is a OT_0-space.

   1.6) An osts (Y, τ_2) is a OT_0-space. If $f : X \to Y$ be injective and ordinary smooth continuous, then $(f^{-1}(Y), \tau_{2f^{-1}(Y)})$ is a OT_0-space.

2) An osts (X, τ) is called a OT_1-*space* if and only if for each $x, y \in X$ with $x \neq y$, there exist $U, V \in S(\tau)$ such that $x \in U, y \notin U$ and $y \in V,\ x \notin V$.

   From the above definitions. I have the following theorems are derived:

2.1) If an osts (X, τ) is a OT_1-space, then (X, τ) is a OT_0-space.

2.2) An osts (X, τ) is a OT_1-space if and only $\overline{\{x\}} = \{x\}$ for every $x \in X$.

2.3) Every subspace of OT_1-space is also OT_1-space.

2.4) Let (X, τ_1) and (Y, τ_2) be two osts and let $f : X \to Y$ be an ordinary smooth homeomorphism. Then (X, τ_1) is an OT_1-space if and only if (Y, τ_2) is an OT_1-space.

2.5) Let $f : X \to Y$ be an injective, ordinary smooth continuous map with respect to the ordinary smooth topologies τ_1 and τ_2 respectively. If (Y, τ_2) is a OT_1-space, then so is (X, τ_1).

2.6) An osts (X, τ_1) is a OT_1-space. If $f : X \to Y$ is an injective and ordinary smooth continuous, then $(f(X), \tau_{2f(X)})$ is a OT_1-space.

2.7) An osts (Y, τ_2) is a OT_1-space. If $f : X \to Y$ is an injective and ordinary smooth open, then $(f^{-1}(Y), \tau_{1f^{-1}(Y)})$ is a OT_1-space.

3) An osts (X, τ) is called a OT_2-*space* if and only if for each $x, y \in X$ with $x \neq y$, there exist $U, V \in S(\tau)$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

From the above definitions. I have the following theorems are derived:

3.1) If an osts (X, τ) is a OT_2-space, then (X, τ) is a OT_1-space.

3.2) If an osts (X, τ) is a OT_1-space, then (X, τ) is not a OT_2-space.

3.3) Every subspace of OT_2-space is also OT_2-space.

3.4) Let (X, τ) be an osts. Then the following conditions are equivalent:

(i) (X, τ) is a OT_2-space.

(ii) Let $p \in X$ for $q \neq p$ there exists $U \in S(\tau), p \in U$ such that $q \notin \overline{U}$.

(iii) For each $p \in X$, $\cap\{\overline{U} : U \in S(\tau), p \in U\} = \{p\}$.

3.5) Let (X, τ_1) and (Y, τ_2) be two osts and let $f : X \to Y$ be an ordinary smooth homeomorphism. Then (X, τ_1) is an OT_2-space if and only if (Y, τ_2) is an OT_2-space.

3.6) Let $f : X \to Y$ be an injective, ordinary smooth continuous map with respect to the ordinary smooth topologies τ_1 and τ_2 respectively. If (Y, τ_2) is a OT_2-space, then so is a (X, τ_1).

3.7) An osts (X, τ) is a OT_2-space. If $f : X \to Y$ is an injective and ordinary smooth open, then $(f(X), \tau_{2f(X)})$ is a OT_2-space.

3.8) An osts (Y, τ_2) is a OT_2-space. If $f : X \to Y$ is an injective and ordinary smooth continuous, then $(f^{-1}(Y), \tau_{1f^{-1}(Y)})$ is a OT_2-space.

4) An osts (X, τ) is called a OT_3-*space* if and only if for each $A \subseteq X$, satisfying $\tau(A^c) > 0$, and each $b \in X$, satisfying $b \notin A$, there exist $U, V \in S(\tau)$ such that $A \subseteq U$, $b \in V$ and $U \cap V = \emptyset$.

From the above definitions. I have the following theorems are derived:

4.1) Every subspace of OT_3-space is also OT_3-space.

4.2) Let (X, τ) be an osts. Then the following conditions are equivalent:

(i) (X, τ) is a OT_3-space.

(ii) For each $x \in X$ and each U containing x, satisfying $\tau(U) > 0$, there exists a set V containing x, satisfying $\tau(V) > 0$ such that $x \in V \subseteq \overline{V} \subseteq U$.

(iii) For each $x \in X$, and each A not containing x, satisfying $\tau(A^c) > 0$, there exists a set V, satisfying $\tau(V) > 0$ containing x such that $\overline{V} \cap A = \emptyset$.

4.3) Let (X, τ_1) and (Y, τ_2) be two osts and let $f : X \to Y$ be an ordinary smooth homeomorphism. Then (X, τ_1) is an OT_3-space if and only if (Y, τ_2) is an OT_3-space.

4.4) Let $f : X \to Y$ be an injective, ordinary smooth closed and ordinary smooth continuous map with respect to the ordinary smooth topologies τ_1 and τ_2 respectively. If (Y, τ_2) is a OT_3-space, then so is a (X, τ_1).

4.5) An osts (X, τ_1) is a OT_3-space. If $f : X \to Y$ is an injective, ordinary smooth continuous and ordinary smooth open, then $(f(X), \tau_{2f(X)})$ is a OT_3-space.

4.6) An osts (Y, τ_2) is a OT_3-space. If $f : X \to Y$ is an injective, ordinary smooth closed and ordinary smooth open, then $(f^{-1}(Y), \tau_{2f^{-1}(Y)})$ is a OT_3-space.

5) An osts (X, τ) is said to be a OT-*regular space* if (X, τ) is a OT_3-space and OT_1-space.

From the above definitions. I have the following theorems are derived:

5.1) Let an osts (X, τ) is a OT-regular space, then (X, τ) is a OT_2-space.

6) An osts (X, τ) is called a OT_4-*space* if and only if for each $A, B \subseteq X$ are disjoint in X, satisfying $\tau(A^c) > 0, \tau(B^c) > 0$, there exist $U, V \in S(\tau)$ such that $A \subseteq U$, $B \subseteq V$ and $U \cap V = \emptyset$.

From the above definitions. I have the following theorems are derived:

6.1) Every subspace of OT_4-space is also OT_4-space.

6.2) Let (X, τ) be an osts. Then the following conditions are equivalent:

(i) (X, τ) is a OT_4-space.

(ii) If $U \subseteq X$, satisfying $\tau(U) > 0$ is a superset of a set A, satisfying $\tau(A^c) > 0$, then there exists set V, satisfying $\tau(V) > 0$ such that $A \subseteq V \subseteq \overline{V} \subseteq U$.

(iii) For each pair of disjoint sets A, B satisfying $\tau(A^c) > 0, \tau(B^c) > 0$, there exists set U, satisfying $\tau(U) > 0$ which $A \subseteq U$ and $\overline{U} \cap B = \emptyset$.

(iv) For each pair of disjoint sets $A, B \subseteq X$, satisfying $\tau(A^c) > 0, \tau(B^c) > 0$, there exist sets $U, V \in S(\tau)$ such that $A \subseteq U, B \subseteq V$ and $\overline{U} \cap \overline{V} = \emptyset$.

6.3) Let (X, τ_1) and (Y, τ_2) be two osts and let $f : X \to Y$ be an ordinary smooth homeomorphism. Then (X, τ_1) is an OT_4-space if and only if (Y, τ_2) is an OT_4-space.

6.4) Let $f : X \to Y$ be an injective, ordinary smooth closed and ordinary smooth continuous map with respect to the ordinary smooth topologies τ_1 and τ_2 respectively. If (Y, τ_2) is a OT_4-space, then so is a (X, τ_1).

6.5) An osts (X, τ_1) is a OT_4-space. If $f : X \to Y$ is an injective, ordinary smooth continuous and ordinary smooth open, then $(f(X), \tau_{2f(X)})$ is a OT_4-space.

6.6) An osts (Y, τ_2) is a OT_4-space. If $f : X \rightarrow Y$ is an injective, ordinary smooth open and ordinary smooth continuous, then $(f^{-1}(Y), \tau_{2f^{-1}(Y)})$ is a OT_4-space.

7) An osts (X, τ) is said to be a OT-*normal space* if (X, τ) is a OT_4-space and OT_1-space.

From the above definitions. I have the following theorems are derived:

7.1) Let an osts (X, τ) is a OT-normal space, then (X, τ) is a OT-regular space.

8) An osts (X, τ) is called a OT_5-*space* if and only if for each A, B are separated sets in X ($\overline{A} \cap B = \emptyset$ and $A \cap \overline{B} = \emptyset$) there exist $U, V \in S(\tau)$ such that $A \subseteq U$, $B \subseteq V$ and $U \cap V = \emptyset$.

From the above definitions. I have the following theorems are derived:

8.1) Let an osts (X, τ) be a OT_5-space, then (X, τ) is a OT_4-space.

8.2) Every subspace of OT_5-space is also OT_5-space.

8.3) Let (X, τ_1) and (Y, τ_2) be two osts and let $f : X \rightarrow Y$ be an ordinary smooth homeomorphism. Then (X, τ_1) is an OT_5-space if and only if (Y, τ_2) is an OT_5-space.

8.4) Let $f : X \rightarrow Y$ be an injective,ordinary smooth closed and ordinary smooth continuous map with respect to the ordinary smooth topologies τ_1 and τ_2 respectively. If (Y, τ_2) is a OT_5-space, then so is a (X, τ_1).

8.5) An osts (X, τ_1) is a OT_5-space. If $f : X \rightarrow Y$ is an injective, ordinary smooth continuous, ordinary smooth closed and ordinary smooth open, then $(f(X), \tau_{2f(X)})$ is a OT_5-space.

8.6) An osts (Y, τ_2) is a OT_5-space. If $f : X \rightarrow Y$ is an injective, ordinary smooth continuous and ordinary smooth open, then $(f^{-1}(Y), \tau_{1f^{-1}(Y)})$ is a OT_5-space.

9) A set A is a OST-*dense set* in X if and only if $\overline{A} = X$.

From the above definitions. I have the following theorems are derived:

9.1) A set A is a OST-dense set in X if and only if $\overset{\circ}{U} \cap A \neq \emptyset$ for all subset U of X with $\overset{\circ}{U} \neq \emptyset$.

9.2) Let (X, τ) is an osts and $A \subseteq X$. Then the following conditions are equivalent:

(i) A is a OST-dense set in X.

(ii) If F is a nonempty subset of X, satisfying $\tau(F^c) > 0$ and $A \subseteq F$, then $F = X$.

(iii) If for all subset U of X with $\overset{\circ}{U} \neq \emptyset$, then $\overset{\circ}{U} \cap A \neq \emptyset$.

9.3) Let X be an osts and A be a subset of X. Then A is OST-dense set in X if and only if $(X \overset{\circ}{\setminus} A) = \emptyset$.

REFERENCES

REFERENCES

[1] Chang C.L. *Fuzzy topological space*. Int. Journal of Math. Analysis and Application 1986; 24: 182-190.

[2] Badard R. *Smooth axiomatics,* . First IFSA Congress,Palma de Maliorca (July 1986).

[3] Ramadan A.A. *Smooth topological spaces* . Fuzzy set and Systems 1992 ; 48: 371-375.

[4] El-gayyar M.K., Kerre E.E., Ramadan A.A. *On smooth topological spaces II separation axioms*. Fuzzy sets and Systems 2001; 119: 495-504.

[5] Pyung K. L., Byeong G.R., Kul H. *Ordinary smooth topological spaces*. Int. Journal of Fuzzy Logic and Intelligent Systems 2012; 1(12): 66-76.

[6] Jeong G.L.,Pyung K.L., Kul H. *Closure, interior redefined and some types of compactness in ordinary smooth topological spaces*. Kor. Journal of Intelligent Systems 2013;1(23): 80-86.

[7] Jeong G.L., Kul H., Pyung K.L. *Closure, interior and compactness in ordinary smooth topological spaces*. Int. Journal of Fuzzy Logic and Intelligent Systems 2014; 3(14): 231-239.

[8] Seymour Lipschutz *Schaum's outline of the theory and problems of general Topology*. McGRAW-HILL book company, New York, St. Louis, san Francisco, Toronto, sydney 1965.

BIOGRAPHY

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