GENERALIZED CLOSED SETS IN GENERALIZED TOPOLOGY AND MINIMAL STRUCTURE SPACES

BY

TATSANEE WIANGWISET

A thesis submitted in partial fulfillment of the requirements for the degree of Master of Science in Mathematics Education at Mahasarakham University

December 2015

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The examining committee has unanimously approved this thesis, submitted by Miss Tatsanee Wiangwiset, as a partial fulfillment of the requirements for the Master of Science degree in Mathematics Education at Mahasarakham University.

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ACKNOWLEDGEMENT

This research was financially supported by Research Support Scholardhip, Mahasarakham University Academic Year 2015.

I wish to express my deepest sincere gratitude to. Asst. Prof. Dr. Chokchai Viriyapong and Dr. Butsakorn Kong-ied for their initial idea, guidance and encouragement which enable me to carry out my study research successfully.

I would like to thank Asst. Prof. Dr. Chawalit Boonpok, Asst. Prof. Dr. Supunnee Sompong and Asst. Prof. Dr. Prapart Pue-on for their constructive comments and suggestions. Sincerely thanks are extended to Dr. Wipawee Tangjai for their valuable comments and suggestion.

I extend my thanks to all the lecturers who have taught me.

I would like to express my sincere gratitude to my parents and my friends who continuously support me.

Finally, I would like to thank all graduate students and staffs at the Department of Mathematics for supporting the preparation of this thesis.

Tatsanee Wiangwiset

ชื่อเรื่อง เซตปิดวางนัยทั่วไปในปริภูมิทอพอโลยีวางนัยทั่วไปและโครงสร้างเล็กสุด
ผู้วิจัย นางสาวทัศนีย์ เวียงวิเศษ
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บทคัดย่อ

งานวิจัยนี้ผู้วิจัยได้นำเสนอแนวคิดของเซตปิด G_μ และเซตปิด G_m ในปริภูมิ GTMS และศึกษาลักษณะเฉพาะบางอย่างของเซตปิดดังกล่าว นอกจากนี้ยังศึกษาแนวคิดของเซตเปิด G_μ และเซตเปิด G_m ในปริภูมิ GTMS มากไปกว่านั้นผู้วิจัยได้นำเสนอแนวคิดของ ปริภูมิ GT_0 -GTMS, ปริภูมิ GT_1 -GTMS, ปริภูมิ GT_2 -GTMS จากการศึกษาเราได้สมบัติพื้นฐานที่สำคัญและความสัมพันธ์ของปริภูมิดังกล่าวกับปริภูมิชนิดอื่น ๆ

คำสำคัญ: ปริภูมิเชิงทอพอโลยีวางนัยทั่วไปและโครงสร้างเล็กสุด ; เซตปิด G_μ ; เซตปิด G_m ; ปริภูมิ GT_1 -GTMS ; ปริภูมิ GT_2 -GTMS ; ปริภูมิ GT_0 -GTMS ; ปริภูมิ GR_o -GTMS ; ปริภูมิ GR_1 -GTMS ; ปริภูมิสมมาตร G -GTMS

TITLE Generalized closed sets in generalized topology and minimal structure spaces

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DEGREE Master of Science **MAJOR** Mathematics Education

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UNIVERSITY Mahasarakham University **YEAR** 2015

ABSTRACT

In this research, we introduce the concepts of G_μ-closed sets and G_m-closed sets in a GTMS space and investigate some of their properties. Moreover, the notion of G_μ-open sets and G_m-open sets in a GTMS space were studied. Furthermore, the concepts of GT_0-GTMS, GT_1-GTMS, GT_2-GTMS were introduced and studied. This research illustrates obtained several properties and their relationships with other types of GTMS spaces.

Keywords: GTMS space; G_μ-closed set; G_m-closed set
GT_1-GTMS space; GT_2-GTMS space; GT_0-GTMS space;
GR_0-GTMS space; GR_1-GTMS space; G-symmetric GTMS space.

CONTENTS

CHAPTER 1

INTRODUCTION

1.1 Background

The concepts of generalized closed sets and generalized open sets in a topological space were first studied by Levine [1]. Also, he studied the properties of these sets together with lower separation axioms in generalized closed sets. In 2000, Popa and Noiri [2] introduced the notion of minimal structure (briefly m-structure). Moreover, they introduced the concepts of m-open sets, m-closed sets m-closure and m-interior. In 2002, Császár [3] introduced the notions of generalized neighborhood systems and generalized topological spaces. He also studied the continuity on generalized neighborhood systems and generalized topological spaces.

In 2011, the concepts of generalized topology and minimal structure spaces (briefly GTMS spaces) and their closed sets were introduced by Buadong, Viriyapong and Boonpok in [4]. Furthermore, they introduced and studied some separation axioms in GTMS spaces which is T_1-GTMS and T_2-GTMS. Later, Zakari [5] introduced the concepts of G-closed and $G*$-closed in GTMS spaces. Moreover, he introduced the concepts of lower separation axioms in GTMS spaces, which is $T_{\frac{1}{2}}$-GTMS and T_0-GTMS, using G-closed sets and $G*$-closed sets, respectively.

In this thesis, we introduce the concepts of G_μ-closed sets and G_m-closed sets in a GTMS space and study some properties of such sets. Moreover, we introduce some separation axioms in the GTMS space using G_μ-open and G_m-open.

1.2 Objective of the research

The purposes of the research are:

1.2.1 To construct and investigate the properties of G_μ-closed and G_m-closed in a generalized topology and minimal structure space.

1.2.2 To construct and investigate some of the separation axioms in generalized topology and minimal structure spaces.

1.3 Research methodology

The research procedure of this thesis consists of the following steps:

1.3.1 Criticism and possible extension of the literature review.

1.3.2 Doing research to investigate the main results.

1.3.3 Applying the results from 1.3.1 and 1.3.2 to the main results.

1.4 Scope of the study

The scopes of the study are: studying some properties of G_μ-closed, G_m-closed and some separation axioms in generalized topology and minimal structure spaces.

CHAPTER 2

PRELIMINARIES

In this chapter, we will give some definitions, notations, dealing with some preliminaries and some useful results that will be duplicated in later chapter.

2.1 Generalized topological spaces

This section discusses some properties of generalized topological space and quasi-topological spaces and some properties of closure and interior.

Definition 2.1.1 [3] Let X be a nonempty set and μ a collection of subsets of X. Then μ is called a generalized topology (briefly GT) on X if and only if $\phi \in \mu$ and any union of elements of μ belongs to μ.

We call the pair (X, μ) a generalized topological space (briefly GTS). The elements of μ are called μ-open sets and the complements of μ-open sets are called μ-closed sets.

Example 1. Let $X = \{1, 2, 3\}$ and $\mu_1 = \{\phi, \{1\}, \{2\}, \{1, 2\}\}$. Then (X, μ_1) is a GTS. It is clear that $\phi, \{1\}, \{2\}, \{1, 2\}$ are μ-open and $X, \{2, 3\}, \{1, 3\}, \{3\}$ are μ-closed in (X, μ_1).

2. Let $X = \{1, 2, 3\}$ and $\mu_2 = \{\phi, \{1\}, \{2\}\}$. Then (X, μ_2) is not a GTS because $\{1\} \in \mu_2$ and $\{2\} \in \mu_2$ but $\{1, 2\} \notin \mu_2$.

Definition 2.1.2 [3] Let X be a nonempty set and μ a generalized topology on X. For a subset A of X, the μ-closure and the μ-interior of A, denoted by $c_\mu(A)$ and $i_\mu(A)$, receptively, are defined as follows :

1. $c_\mu(A) = \cap\{F \mid A \subset F, X \setminus F \in \mu\}$,
2. $i_\mu(A) = \cup\{G \mid G \subset A, G \in \mu\}$.

Theorem 2.1.3 [3] Let (X,μ) be a generalized topological space and $A \subset X$. Then

1. $c_\mu(A) = X \setminus i_\mu(X \setminus A)$,
2. $i_\mu(A) = X \setminus c_\mu(X \setminus A)$.

Proposition 2.1.4 [3] Let (X,μ) be a generalized topological space and $A \subset X$. Then

1. $x \in i_\mu(A)$ if and only if there exists $V \in \mu$ such that $x \in V \subset A$,
2. $x \in c_\mu(A)$ if and only if $V \cap A \neq \phi$ for every μ-open set V containing x.

Proposition 2.1.5 [7] Let (X,μ) be a generalized topological space. For subsets A and B of X, the following properties hold.

1. $c_\mu(X \setminus A) = X \setminus i_\mu(A)$ and $i_\mu(X \setminus A) = X \setminus c_\mu(A)$,
2. if $X \setminus A \in \mu$, then $c_\mu(A) = A$ and if $A \in \mu$, then $i_\mu(A) = A$,
3. if $A \subset B$, then $c_\mu(A) \subset c_\mu(B)$ and $i_\mu(A) \subset i_\mu(B)$,
4. $A \subset c_\mu(A)$ and $i_\mu(A) \subset A$,
5. $c_\mu(c_\mu(A)) = c_\mu(A)$ and $i_\mu(i_\mu(A)) = i_\mu(A)$.

Definition 2.1.6 [8] A generalized topological space (X,μ) is a quasi-topological space if and only if $U \in \mu$ and $V \in \mu$ implies $U \cap V \in \mu$. In this case, μ is called a quasi-topology on X.

Lemma 2.1.7 [9] If (X,μ) is a quasi-topological space, then the following hold.

1. If A and B are μ-open sets, then $A \cap B$ is a μ-open set.
2. $i_\mu(A \cap B) = i_\mu(A) \cap i_\mu(B)$ for every subsets A and B of X.
3. $c_\mu(A \cup B) = c_\mu(A) \cup c_\mu(B)$ for every subsets A and B of X.

2.2 Minimal structure spaces

Minimal structure and some properties of closure and interior are discussed in this section.

Definition 2.2.1 [2] A subfamily m of the power set P(X) of a nonempty set X is called a minimal structure (briefly m-structure) on X if $\phi \in m$ and $X \in m$. By (X, m), we denote a nonempty set X with a minimal structure m on X and call it an m-space. Each member of m is said to be m-open and the complement of an m-open set is said to be m-closed.

Definition 2.2.2 [2] Let (X, m) be a m-space. For a subset A of X, the c_m of A and i_m of A are defined in as follows:

1. $c_m(A) = \cap\{F : A \subset F, X \setminus F \in m\}$
2. $i_m(A) = \cup\{U : U \subset A, U \in m\}$

Lemma 2.2.3 [10] Let (X, m) be a nonempty set and m a minimal structure on X. For subsets A and B of X, the following properties hold.

1. $c_m(X \setminus A) = X \setminus i_m(A)$ and $i_m(X \setminus A) = X \setminus c_m(A)$,
2. if $X \setminus A \in m$, then $c_m(A) = A$ and if $A \in m$, then $i_m(A) = A$,
3. $c_m(\phi) = \phi$, $c_m(X) = X$, $i_m(\phi) = \phi$ and $i_m(X) = X$,
4. if $A \subset B$, then $c_m(A) \subset c_m(B)$ and $i_m(A) \subset i_m(B)$,
5. $A \subset c_m(A)$ and $i_m(A) \subset A$,
6. $c_m(c_m(A)) = c_m(A)$ and $i_m(i_m(A)) = i_m(A)$.

Definition 2.2.4 [11] A minimal structure m on a nonempty set X is said to have property B if the union of any family of subsets belonging to m belongs to m.

Lemma 2.2.5 [2] For a minimal structure m on a nonempty set X, the following are equivalent:

1. m has the property B;
2. If $m \setminus i_m(V) = V$, then $V \in m$;
3. If $m \setminus c_m(F) = F$, then $X \setminus F \in m$.

2.3 Generalized topology and minimal structure spaces

This section compiles some properties of generalized topology and minimal structure space and some properties of closure and interior. Moreover, some properties s-closed and c-closed and some separation axioms in GTMS space are also included.

Definition 2.3.1 [4] Let X be a nonempty set and let μ be a generalized topology and m a minimal structure on X. A triple (X,μ,m) is called a generalized topology and minimal structure space (briefly GTMS space).

Let (X,μ,m) be a generalized topology and minimal structure space and A a subset of X. The closure and interior of A in μ are denote by $c_\mu(A)$ and $i_\mu(A)$, respectively. And the closure and interior of A in m are denote by $c_m(A)$ and $i_m(A)$, respectively.

Definition 2.3.2 [4] Let (X,μ,m) a GTMS space. A subset A of X is said to be a μm-closed set if $c_\mu(c_m(A)) = A$. And a subset A of X is said to be a $m\mu$-closed set if $c_m(c_\mu(A)) = A$. The complement of a μm-closed (resp. $m\mu$-closed) set is said to be μm-open (resp. $m\mu$-open).

Lemma 2.3.3 [4] Let (X,μ,m) be a GTMS space and $A \subset X$. Then A is μm-closed if and only if $c_m(A) = A$ and $c_\mu(A) = A$.

Lemma 2.3.4 [4] Let (X,μ,m) be a GTMS space and $A \subset X$. Then A is $m\mu$-closed if and only if $c_m(A) = A$ and $c_\mu(A) = A$.

Proposition 2.3.5 [4] Let (X,μ,m) be a GTMS space and $A \subset X$. Then A is μm-closed if and only if A is $m\mu$-closed.

Definition 2.3.6 [4] Let (X,μ,m) be a GTMS space and A a subset of X. Then A is said to be closed if A is μm-closed. The complement of closed set is said to be an open set.

Remark 2.3.7 [4] Let (X,μ,m) be a GTMS space and A a subset of X. Then A is closed if and only if A is $m\mu$-closed.

Proposition 2.3.8 [4] Let (X,μ,m) be a GTMS space. If A and B are closed, then $A \cap B$ is closed.

Remark 2.3.9 [4] The union of two closed sets is not a closed set in general as can be seen from the following example.

Example Let $X = \{a, b, c, d\}$. We define generalized topology μ and minimal structure m on X as follows: $\mu = \{\phi, \{c, d\}, \{a, c, d\}, \{b, c, d\}, X\}$ and $m = \{\phi, \{a, c, d\}, \{b, c, d\}, X\}$. Then $\{a\}$ and $\{b\}$ are closed but $\{a\} \cup \{b\} = \{a, b\}$ is not closed.

Proposition 2.3.10 [4] Let (X,μ,m) be a GTMS space. Then A is open if and only if $A = i_\mu(i_m(A))$.

Proposition 2.3.11 [4] Let (X,μ,m) be a GTMS space. If A and B are open, then $A \cup B$ is open.

Remark 2.3.12 [4] The intersection of two open sets is not a open set in general as can be seen from the following example.

Example Let $X=\{a, b, c\}$. We define generalized topology μ and minimal structure m on X as follows: $\mu=\{\phi, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$, and $m=\{\phi, \{a\}, \{a, b\}, \{a, c\}, X\}$. Then $\{a,b\}$ and $\{a,c\}$ are open but $\{a,b\}\cap\{a,c\}=\{a\}$ is not open.

Definition 2.3.13 [4] Let (X,μ,m) be a GTMS space and A a subset of X. Then A is said to be s-closed if $c_\mu(A)=c_m(A)$. And A is said to be c-closed if $c_\mu(c_m(A))=c_m(c_\mu(A))$. The complement of a s-closed (resp. c-closed) set is called a s-open (resp. c-open) set.

Proposition 2.3.14 [4] Let (X,μ,m) be a GTMS space and $A\subset X$. If A is closed, then A is s-closed.

Remark 2.3.15 [4] The converse of Proposition 2.3.14 is not true. We can be seen from the following example.

Example Let $X=\{a, b, c, d\}$. We define generalized topology μ and minimal structure m on X as follows: $\mu=\{\phi, \{a\}, \{a, c\}\}$ and $m=\{\phi, \{a\}, \{b, c\}, X\}$. Then $c_\mu(\{c\})=\{b, c\}=c_m(\{c\})$. But $c_\mu(c_m(\{c\}))=\{b, c\}\neq\{c\}$.

Proposition 2.3.16 [4] Let (X,μ,m) be a GTMS space and $A\subset X$. If A is s-closed, then A is c-closed.

Remark 2.3.17 [4] The converse of Proposition 2.3.16 is not true. We can be seen from the following example.

Example Let $X=\{a, b, c\}$. We define generalized topology μ and minimal structure m on X as follows: $\mu=\{\phi, \{a\}, \{a, c\}\}$ and $m=\{\phi, \{b\}, \{b, c\}, X\}$. Then $c_\mu(c_m(\{c\}))=X=c_m(c_\mu(\{c\}))$. But $c_\mu(\{c\})=\{b, c\}\neq\{a, c\}=c_m(\{c\})$.

Proposition 2.3.18 [4] Let (X,μ,m) be a GTMS space and $A \subset X$. Then

1. A is s-open if and only if $i_\mu(A) = i_m(A)$,
2. A is c-open if and only if $i_\mu(i_m(A)) = i_m(i_\mu(A))$.

Proposition 2.3.19 [4] Let (X,μ,m) be a GTMS space and $A \subset X$. If A is open, then A is s-open.

Proposition 2.3.20 [4] Let (X,μ,m) be a GTMS space and $A \subset X$. If A is s-open, then A is c-open.

Lemma 2.3.21 [4] Let (X,μ,m) be a GTMS space and $A \subset X$. If A is s-closed, then $c_\mu(A)$ is closed.

Remark 2.3.22 [4] Let (X,μ,m) be a GTMS space and $A \subset X$. If A is s-closed, then $c_m(A)$ is closed.

Theorem 2.3.23 [4] Let (X,μ,m) be a GTMS space and $A \subset X$. Then A is closed if and only if there exists a s-closed set B such that $c_\mu(B) = A$.

Lemma 2.3.24 [4] Let (X,μ,m) be a GTMS space and $A \subset X$. If A is c-closed, then $c_\mu(c_m(A))$ is closed.

Remark 2.3.25 [4] Let (X,μ,m) be a GTMS space and $A \subset X$. If A is c-closed, then $c_m(c_\mu(A))$ is closed.

Theorem 2.3.26 [4] Let (X,μ,m) be a GTMS space and $A \subset X$. Then A is closed if and only if there exists a c-closed set B such that $A = c_\mu(c_m(B))$.

To be discussed some properties of G-closed and $G*$-closed. Moreover, we will some discuss properties and relations of T_1-GTMS, T_0-GTMS and R_0-GTMS in [5].

Definition 2.3.27 [5] Let (X,μ,m) be a GTMS space. A subset A of X is said to be μmG-closed if $c_\mu(c_m(A))\subset U$ whenever $A\subset U$ and U is open. A subset A of X is said to be $m\mu G$-closed if $c_m(c_\mu(A))\subset U$ whenever $A\subset U$ and U is open. A subset A of X is said to be G-closed if A is $m\mu G$-closed and μmG-closed.

Theorem 2.3.28 [5] Let (X,μ,m) be a GTMS space. Then every closed set is G-closed.

Definition 2.3.29 [5] Let (X,μ,m) be a GTMS space. A subset A of X is said to be μG-closed if $c_m(c_\mu(A))\subset U$ whenever $A\subset U$ and U is μ-open. A subset A of X is said to be mG-closed if $c_\mu(c_m(A))\subset U$ whenever $A\subset U$ and U is m-open. A subset A of X is said to be $G*$-closed if A is μG-closed and mG-closed.

Theorem 2.3.30 [5] Let (X,μ,m) be a GTMS space. Then every closed set is $G*$-closed.

Theorem 2.3.31 [5] Let (X,μ,m) be a GTMS space and $A\subset X$. Then the following hold.

1. If A is μmG-closed and F is closed, then $A\cap F$ is μmG-closed.
2. If A is $m\mu G$-closed and F is closed, then $A\cap F$ is $m\mu G$-closed.
3. If A is μG-closed and F is closed, then $A\cap F$ is μG-closed.

Corollary 2.3.32 [5] Let (X,μ,m) be a GTMS space and $A\subset X$. If A is G-closed and F is closed, then $A\cap F$ is G-closed.

Theorem 2.3.33 [5] Let (X,μ,m) be a GTMS space and $A\subset X$. Then the following hold.

1. If A is open and μmG-closed, then A is closed.
2. If A is open and $m\mu G$-closed, then A is closed.
3. If A is μ-open and μG-closed, then A is closed.
4. If A is m-open and mG-closed, then A is closed.

Corollary 2.3.34 [5] Let (X,μ,m) be a GTMS space and $A\subset X$. Then the following hold.

1. If A is open and G-closed, then A is closed.
2. If A is open and G^*-closed, then A is closed.

Theorem 2.3.35 [5] Let (X,μ,m) be a GTMS space and $A\subset X$. Then the following hold.

1. If A is μmG-closed, then $c_\mu(c_m(A))\setminus A$ does not contain any nonempty closed set.
2. If A is $m\mu G$-closed, then $c_m(c_\mu(A))\setminus A$ does not contain any nonempty closed set.
3. If A is μG-closed, then $c_m(c_\mu(A))\setminus A$ does not contain any nonempty μ-closed set.
4. If A is mG-closed, then $c_\mu(c_m(A))\setminus A$ does not contain any nonempty m-closed set.

Corollary 2.3.36 [5] Let (X,μ,m) be a GTMS space and $A\subset X$. Then the following hold.

1. If A is G-closed, then $c_\mu(c_m(A))\setminus A$ and $c_m(c_\mu(A))\setminus A$ do not contain any nonempty closed set.
2. If A is G^*-closed, then $c_m(c_\mu(A))\setminus A$ does not contain any nonempty μ-closed set and $c_\mu(c_m(A))\setminus A$ does not contain any nonempty m-closed set.

Theorem 2.3.37 [5] Let (X,μ,m) be a GTMS space and $A \subset X$. Then the following hold.

1. If A is μmG-closed and $A \subset B \subset c_m(A)$, then B is μmG-closed.
2. If A is $m\mu G$-closed and $A \subset B \subset c_\mu(A)$, then B is $m\mu G$-closed.
3. If A is μG-closed and $A \subset B \subset c_\mu(A)$, then B is μG-closed.
4. If A is mG-closed and $A \subset B \subset c_m(A)$, then B is mG-closed.
5. If A is μmG-closed and $c_\mu(B) \subset A \subset B$, then B is $m\mu G$-closed.
6. If A is $m\mu G$-closed and $c_m(B) \subset A \subset B$, then B is μmG-closed.

The following section explains T_1-GTMS space, T_2-GTMS space and R_0-GTMS space. Also, the section discusses some separation axioms.

Definition 2.3.38 [4] A GTMS space (X,μ,m) is called a T_1-GTMS space if for any pair of distinct points x and y in X, there exist a μ-open set U and a m-open set V such that $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$.

Definition 2.3.39 [4] A GTMS space (X,μ,m) is called a Hausdorff GTMS or T_2-GTMS space if for any pair of distinct points x and y in X, there exist a μ-open set U and a m-open set V such that $x \in U$, $y \in V$ and $U \cap V = \phi$.

Proposition 2.3.40 [4] Let (X,μ,m) be a GTMS space. If X is T_2-GTMS space, then X is T_1-GTMS space.

Remark 2.3.41 [4] The converse of Proposition 2.3.40 is not true. We can be seen from the following example.

Example 2.3.42 [4] Let $X=\{a,b,c\}$. We define generalized topology μ and minimal structure space μ on X as follow: $\mu=\{\phi,\{a\},\{c\},\{a,b\},\{a,c\},\{b,c\},X\}$ and $m=\{\phi,\{a\},\{a,b\},\{a,c\},\{b,c\},X\}$. Then X is a T_1-GTMS space. But X is not a T_2-GTMS space.

Theorem 2.3.43 [4] Let (X,μ,m) be a GTMS space. Then the following are equivalent:

1. X is a T_2-GTMS space.
2. If $x\in X$, then for each $y\neq x$, there exists a μ-open set U containing x such that $y\notin c_m(U)$
3. For each $x\in X$, $\{x\}=\cap\{c_m(U):U\in\mu$ and $x\in U\}$.

Theorem 2.3.44 [4] Let (X,μ,m) be a GTMS space. Then the following are equivalent:

1. X is a T_2-GTMS space.
2. If $x\in X$, then for each $y\neq x$, there exists a m-open set V containing x such that $y\notin c_\mu(U)$
3. For each $x\in X$, $\{x\}=\cap\{c_\mu(V):V\in\mu$ and $x\in V\}$.

Definition 2.3.45 [5] A GTMS space (X,μ,m) is called a T_0-GTMS space if for any pair of distinct points x and y in X, there exist a subset U which is either μ-open or m-open such that $x\in U$, $y\notin U$ or $y\in U$, $x\notin U$.

Lemma 2.3.46 [5] A GTMS space (X,μ,m) is a T_0-GTMS space if and only if for any pair of distinct points x and y in X, $c_\mu(\{x\})\neq c_\mu(\{y\})$ or $c_m(\{x\})\neq c_m(\{y\})$.

Definition 2.3.47 [5] A GTMS space (X,μ,m) is called a R_0-GTMS space if $\{x\}$ is G^*-closed set for each $x\in X$.

Theorem 2.3.48 [5] Let (X,μ,m) be a GTMS space. Then the following are equivalent.

1. X is a R_0-GTMS space.

2. For each $x,y\in X$, if $x\notin c_\mu(\{y\})$, then $y\notin c_m(c_\mu(\{x\}))$ and if $x\notin c_m(\{y\})$, then $y\notin c_\mu(c_m(\{x\}))$.

3. For each $x,y\in X$, if $x\in c_m(c_\mu(\{y\}))$, then $y\in c_\mu(\{x\})$ and if $x\in c_\mu(c_m(\{y\}))$, then $y\in c_m(\{x\})$.

Theorem 2.3.49 [5] Let (X,μ,m) be a R_0-GTMS space. Then for each $x,y\in X$,

$c_\mu(\{x\})=c_\mu(\{y\})$ or $c_\mu(\{x\})\cap c_\mu(\{y\})=\phi$, also $c_m(\{x\})=c_m(\{y\})$ or $c_m(\{x\})\cap c_m(\{y\})=\phi$.

Theorem 2.3.50 [5] Let (X,μ,m) be a GTMS space. Then the following are equivalent.

1. X is a T_1-GTMS space.

2. X is a T_0-GTMS space and R_0-GTMS space.

CHAPTER 3

G_μ-closed set and G_m-closed set

In this chapter, we will introduce the notions of G_μ-closed sets and G_m-closed sets in a GTMS spaces.

3.1 G_μ-closed set

In this section, we will introduce the notion of G_μ-closed set and investigate some of their properties.

Definition 3.1.1 Let (X,μ,m) be a GTMS space. A subset A of X is said to be a G_μ-closed set if $c_\mu(A)\subset U$ whenever $A\subset U$ and U is open.

Example 3.1.2 Let $X=\{1,2,3,4\}$ with generalized topology
$\mu=\{\phi,\{2,3\},\{2,4\},\{2,3,4\}\}$ and minimal structure $m=\{\phi,\{1,3\},\{2,4\},\{3,4\},X\}$.
Then $\phi,\{2,4\},\{2,3,4\}$ are open. Hence $\{1\},\{1,2\},\{1,3\},\{1,4\},\{1,2,3\},\{1,2,4\},$
$\{1,3,4\},X$ are G_μ-closed.

Proposition 3.1.3 Let (X,μ,m) be a GTMS space and $A\subset X$. If A is μ-closed, then A is G_μ-closed.

Proof. Assume that A is μ-closed. To show that A is G_μ-closed, let U be an open set such that $A\subset U$. Since A is μ-closed, $c_\mu(A)=A$. This implies $c_\mu(A)\subset U$. Hence A is G_μ-closed.

Proposition 3.1.4 Every μmG-closed set in a GTMS space (X,μ,m) is G_μ-closed.

Proof. Assume that A is μmG-closed. To show that A is G_μ-closed, let U be open such that $A\subset U$. Since A is μmG-closed, $c_\mu(c_m(A))\subset U$.

Since $c_\mu(A)\subset c_\mu(c_m(A))$, $c_\mu(A)\subset U$. Therefore A is G_μ-closed.

Remark The converse of the previous proposition need not be true as the following example.

Example 3.1.5 Let $X=\{1,2,3,4\}$ with generalized topology $\mu=\{\phi,\{3,4\},\{1,2,3\},X\}$ and minimal structure $m=\{\phi,\{2,4\},\{1,2,3\},X\}$. Then $\phi,\{1,2,3\},X$ are open. Hence $\phi,\{4\},\{1,4\},\{2,4\},\{3,4\},\{1,2,4\},\{1,3,4\},\{2,3,4\},X$ are μmG-closed and $\phi,\{1\},\{2\},\{4\},\{1,2\}$ $\{1,4\},\{2,4\},\{3,4\},\{1,2,4\},\{1,3,4\},\{2,3,4\},X$ are G_μ-closed. Then $\{1,2\}$ is G_μ-closed but $\{1,2\}$ is not μmG-closed.

Proposition 3.1.6 Every $m\mu G$-closed set in a GTMS space (X,μ,m) is G_μ-closed.

Proof. Assume that A is $m\mu G$-closed. To show that A is G_μ-closed, let U be open such that $A\subset U$. Since A is $m\mu G$-closed, $c_m(c_\mu(A))\subset U$.

Since $c_\mu(A)\subset c_m(c_\mu(A))$, $c_\mu(A)\subset U$. Therefore A is G_μ-closed.

Remark The converse of the previous proposition need not be true as the following example.

Example 3.1.7 Let $X=\{1,2,3,4\}$ with generalized topology $\mu=\{\phi,\{3,4\},\{1,2,3\},X\}$ and minimal structure $m=\{\phi,\{2,4\},\{1,2,3\},X\}$. Then $\phi,\{1,2,3\},X$ are open. Hence $\phi,\{4\},\{1,4\},\{2,4\},\{3,4\},\{1,2,4\},\{1,3,4\},\{2,3,4\},X$ are $m\mu G$-closed and $\phi,\{1\},\{2\},\{4\},\{1,2\}$

$\{1,4\},\{2,4\},\{3,4\},\{1,2,4\},\{1,3,4\},\{2,3,4\},X$ are G_μ-closed. Then $\{2\}$ is G_μ-closed but $\{2\}$ is not $m\mu G$-closed.

Theorem 3.1.8 Let (X,μ,m) be a GTMS space and $A,F\subset X$. If A is G_μ-closed and F is closed, then $A\cap F$ is G_μ-closed.

Proof. Assume that A is G_μ-closed and F is closed. To show that $A\cap F$ is G_μ-closed, let V be an open set such that $A\cap F\subset V$. Then $A\subset V\cup(X\setminus F)$ and $V\cup(X\setminus F)$ is open. Since A is G_μ-closed, $c_\mu(A)\subset V\cup(X\setminus F)$.

Thus $c_\mu(A)\cap F\subset V$. This implies $c_\mu(A\cap F)\subset V$. Hence $A\cap F$ is G_μ-closed.

Proposition 3.1.9 Let (X,μ,m) be a GTMS space and $A\subset X$. If A is open and G_μ-closed, then A is μ-closed.

Proof. Assume that A is open and G_μ-closed. Since $A\subset A$, $c_\mu(A)\subset A$. But $A\subset c_\mu(A)$, $c_\mu(A)=A$. Thus A is μ-closed.

Example 3.1.10 Let $X=\{1,2,3,4\}$ with generalized topology $\mu=\{\phi,\{2,3\},\{2,4\},\{2,3,4\}\}$ and minimal structure $m=\{\phi,\{1,3\},\{2,4\},\{3,4\},X\}$.

Then $\phi,\{2,4\},\{2,3,4\}$ are open. Hence $\{1\},\{1,2\},\{1,3\},\{1,4\},\{1,2,3\},\{1,2,4\},\{1,3,4\},X$ are G_μ-closed. But $\{1,2\},\{1,2,3\},\{1,2,4\},\{1,3,4\}$ is not μ-closed.

Proposition 3.1.11 Let (X,μ,m) be a GTMS space and $A\subset X$. If A is G_μ-closed, then $c_\mu(A)\setminus A$ does not contain any nonempty closed set.

Proof. Assume that A is G_μ-closed. Suppose to the contrary that $c_\mu(A)\setminus A$ contains a nonempty closed set, say F. Then $F\subset c_\mu(A)\setminus A=c_\mu(A)\cap(X\setminus A)$. Thus $F\subset X\setminus A$, and so $A\subset X\setminus F$ Moreover, $X\setminus F$ is open. Since A is G_μ-closed, $c_\mu(A)\subset X\setminus F$. This implies $F\subset X\setminus c_\mu(A)$. From $F\subset c_\mu(A)$, $F=\phi$ which contradicts with $F\neq\phi$.

Lemma 3.1.12 Let (X,μ,m) be a GTMS space and $A\subset X$. If A is s-closed, then $c_\mu(c_m(A))=c_\mu(A)=c_m(A)=c_m(c_\mu(A))$.

Proof. Assume that A is s-closed. Then $c_\mu(A)=c_m(A)$.

Hence $c_\mu(c_m(A))=c_\mu(c_\mu(A))=c_\mu(A)=c_m(A)=c_m(c_m(A))=c_m(c_\mu(A))$.

Theorem 3.1.13 Let (X,μ,m) be a GTMS space and $A\subset X$. If A is G_μ-closed and s-closed, then A is G-closed.

Proof. Assume that A is G_μ-closed and s-closed. To show that A is G-closed, let U be open such that $A\subset U$. Since A is G_μ-closed, $c_\mu(A)\subset U$. Since A is s-closed, by Lemma 3.1.12, $c_\mu(A)=c_m(c_\mu(A))=c_\mu(c_m(A))$. Then $c_\mu(c_m(A))\subset U$ and $c_m(c_\mu(A))\subset U$. Hence A is G-closed.

Definition 3.1.14 Let (X,μ,m) be a GTMS space. A subset A of X is said to be a G_μ-open set if $X\setminus A$ is G_μ-closed.

Theorem 3.1.15 Let (X,μ,m) be a GTMS space and $A\subset X$. Then A is G_μ-open if and only if $F\subset i_\mu(A)$ whenever F is closed and $F\subset A$.

Proof. ($\rightarrow$) Assume that A is G_μ-open. Let F be closed such that $F\subset A$. Then $X\setminus F$ is open and $X\setminus A\subset X\setminus F$. Since A is G_μ-open, thus $X\setminus A$ is G_μ-closed and so $c_\mu(X\setminus A)\subset X\setminus F$. This implies $X\setminus i_\mu(A)\subset X\setminus F$. Hence $F\subset i_\mu(A)$.

($\leftarrow$) Assume that $F\subset i_\mu(A)$ whenever F is closed and $F\subset A$. To show that A is G_μ-open, let U be open and $X\setminus A\subset U$. Then $X\setminus U$ is closed and $X\setminus U\subset A$. By assumption, $X\setminus U\subset i_\mu(A)$. Thus $X\setminus i_\mu(A)\subset U$, and so $c_\mu(X\setminus A)\subset U$. Hence $X\setminus A$ is G_μ-closed, and so A is G_μ-open.

Theorem 3.1.16 Let (X,μ,m) be a GTMS space and $A \subset X$. If A is G_μ-closed, then $c_\mu(A) \setminus A$ is G_μ-open.

Proof. Assume that A is G_μ-closed. Suppose to the contrary that $c_\mu(A) \setminus A$ is not G_μ-open. By Theorem 3.1.15, there exists a closed set F such that $F \subset c_\mu(A) \setminus A$ and $F \not\subset i_\mu(c_\mu(A) \setminus A)$. Since $F \not\subset i_\mu(c_\mu(A) \setminus A)$, $F \neq \phi$. It is a contradiction with Proposition 3.1.11. Hence $c_\mu(A) \setminus A$ is G_μ-open.

Proposition 3.1.17 Let (X,μ,m) be a GTMS space and $A \subset X$. If A is G_μ-open and $i_\mu(A) \subset B \subset A$, then B is G_μ-open.

Proof. Assume that A is G_μ-open and $i_\mu(A) \subset B \subset A$. To show that B is G_μ-open, let V be closed such that $V \subset B$. Since $V \subset B$ and $B \subset A$, $V \subset A$. From A is G_μ-open and V is closed which $V \subset A$, By Theorem 3.1.15, $V \subset i_\mu(A)$. Since $i_\mu(A) \subset B$, $i_\mu(i_\mu(A)) \subset i_\mu(B)$, and so $i_\mu(A) \subset i_\mu(B)$. Hence $V \subset i_\mu(B)$. Then B is G_μ-open.

Theorem 3.1.18 Let (X,μ,m) be a GTMS space and $A \subset X$. If A is G_μ-open, then $U = X$ whenever U is open and $i_\mu(A) \cup (X \setminus A) \subset U$.

Proof. Assume that A is G_μ-open. Suppose to the contrary that there exists an open set U such that $i_\mu(A) \cup (X \setminus A) \subset U$ and $U \neq X$. Then $X \setminus U \neq \phi$ and $X \setminus U$ is closed. Since $i_\mu(A) \cup (X \setminus A) \subset U$, $X \setminus U \subset X \setminus (i_\mu(A) \cup (X \setminus A))$. Thus $X \setminus U \subset c_\mu(X \setminus A) \cap A$, and so $X \setminus U \subset c_\mu(X \setminus A) \setminus (X \setminus A)$. It is a contradiction with $X \setminus A$ is G_μ-closed. Hence $U = X$.

Theorem 3.1.19 Let (X,μ,m) be a GTMS space such that μ is a QT on X. If A and B are G_μ-closed, then $A \cup B$ is G_μ-closed.

Proof. Assume that A and B are G_μ-closed and let U be an open set such that $A \cup B \subset U$. Then $A \subset U$ and $B \subset U$. Thus $c_\mu(A) \subset U$ and $c_\mu(B) \subset U$, and so $c_\mu(A) \cup c_\mu(B) \subset U$. Since μ is a QT, $c_\mu(A \cup B) \subset U$. Hence $A \cup$ B is G_μ-closed.

Theorem 3.1.20 Let (X, μ, m) be a GTMS space with $X \notin \mu$ and $A, B \subset X$. If A is G_μ-closed $A \subset B$, then B is G_μ-closed.

Proof. Assume that A is G_μ-closed and $A \subset B$. Suppose B is not G_μ-closed. Thus there exists an open set U such that $B \subset U$ and $c_\mu(B) \not\subset U$. Since A is G_μ-closed, $c_\mu(A) \subset U$ and $X \setminus U \subset X \setminus c_\mu(A)$. Then $X = (X \setminus U) \cup U \subset (X \setminus c_\mu(A)) \cup U \subset X$ is μ-open which contradicts with $X \notin \mu$. Thus B is G_μ-closed.

Corollary 3.1.21 Let (X, μ, m) be a GTMS space with $X \notin \mu$. If A or B are G_μ-closed, then $A \cup B$ is G_μ-closed.

Proof. Assume A or B are G_μ-closed. Without loss of generality, we assume that A is G_μ-closed. Since $A \subset A \cup B$, by Theorem 3.1.20 , $A \cup B$ are G_μ-closed.

3.2 G_m-closed set

In this section, we will introduce the notion of G_m-closed set and investigate some of their properties.

Definition 3.2.1 Let (X, μ, m) be a GTMS space. A subset A of X is said to be a G_m-closed set if $c_m(A) \subset U$ whenever $A \subset U$ and U is open.

Example 3.2.2 Let $X = \{1, 2, 3, 4\}$ with generalized topology $\mu = \{\phi, \{2,4\}, \{1,4\}, \{1,2,4\}\}$ and minimal structure $m = \{\phi, \{3\}, \{2,4\}, \{1,2,3\}, X\}$. Then

$\phi,\{2,4\}$ are open. Hence $\phi,\{1\},\{3\},\{4\},\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{3,4\},\{1,2,3\},\{1,2,4\}$ $\{1,3,4\},\{2,3,4\}$ are G_m -closed.

Proposition 3.2.3 Let (X,μ,m) be a GTMS space and $A\subset X$. If A is m -closed, then A is G_m -closed.

Proof. Assume that A is m -closed. To show that A is G_m -closed, let U be open such that $A\subset U$. Since A is m -closed, $c_m(A)=A$. This implies $c_m(A)\subset U$. Hence A is G_m -closed.

Proposition 3.2.4 Every μmG -closed set in a GTMS space (X,μ,m) is G_m -closed.

Proof. Assume that A is μmG -closed. To show that A is G_m -closed, let U be open, such that $A\subset U$. Since A is μmG -closed, that $c_\mu(c_m(A))\subset U$.

Since $c_m(A)\subset c_\mu(c_m(A))$, $c_m(A)\subset U$. Therefore A is G_m -closed.

Remark The converse of the previous proposition need not be true as the following example.

Example 3.2.5 Let $X=\{1,2,3,4\}$ with generalized topology $\mu=\{\phi,\{3,4\},\{1,2,3\},X\}$ and minimal structure $m=\{\phi,\{2,4\},\{1,2,3\},X\}$. Then $\phi,\{1,2,3\},X$ are open. Hence $\phi,\{4\},\{1,4\},\{2,4\},\{3,4\},\{1,2,4\},\{1,3,4\},\{2,3,4\},X$ are μmG -closed and $\phi,\{1\},\{3\},\{4\},\{1,3\},\{1,4\},\{2,4\},\{3,4\},\{1,2,4\},\{1,3,4\},\{2,3,4\},X$ are G_m -closed. Then $\{3\}$ is G_m -closed but $\{3\}$ is not μmG -closed.

Proposition 3.2.6 Every $m\mu G$ -closed set in a GTMS space (X,μ,m) is G_m -closed.

Proof. Assume that A is $m\mu G$ -closed. To show that A is G_m -closed, let U be open, such that $A \subset U$. Since A is $m\mu G$-closed, $c_m(c_\mu(A)) \subset U$.

Since $c_m(A) \subset c_m(c_\mu(A))$, $c_m(A) \subset U$. Therefore A is G_m-closed.

Remark The converse of the previous proposition need not be true as the following example.

Example 3.2.7 Let $X = \{1,2,3,4\}$ with generalized topology $\mu = \{\phi, \{3,4\}, \{1,2,3\}, X\}$ and minimal structure $m = \{\phi, \{2,4\}, \{1,2,3\}, X\}$. Then $\phi, \{1,2,3\}, X$ are open. Hence $\phi, \{4\}, \{1,4\}, \{2,4\}, \{3,4\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}, X$ are $m\mu G$-closed and $\phi, \{1\}, \{3\}, \{4\}, \{1,3\}, \{1,4\}, \{2,4\}, \{3,4\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}, X$ are G_m-closed. Then $\{1,3\}$ is G_m-closed but $\{1,3\}$ is not $m\mu G$-closed.

Theorem 3.2.8 Let (X, μ, m) be a GTMS space and $A, F \subset X$. If A is G_m-closed and F is closed, then $A \cap F$ is G_m-closed.

Proof. Assume that A is G_m-closed and F is closed. To show that $A \cap F$ is G_m-closed, let V be an open set such that $A \cap F \subset V$. Then $A \subset V \cup (X \setminus F)$ and $V \cup (X \setminus F)$ is open. Since A is G_m-closed, $c_m(A) \subset V \cup (X \setminus F)$.

Thus $c_m(A) \cap F \subset V$. This implies $c_m(A \cap F) \subset V$. Hence $A \cap F$ is G_m-closed.

Proposition 3.2.9 Let (X, μ, m) be a GTMS space and $A \subset X$. If A is open and G_m-closed and m has the property B, then A is m-closed.

Proof. Assume that A is open and G_m-closed and m has the property B. Since A is open and G_m-closed, $c_m(A) \subset A$. Since $A \subset c_m(A)$, $c_m(A) = A$. From m has the property B, A is m-closed.

Theorem 3.2.10 Let (X, μ, m) be a GTMS space and $A \subset X$. If A is G_m-closed and s-closed, then A is G-closed.

Proof. Assume that A is G_m-closed and s-closed. To show that A is G-closed, let U be open such that $A \subset U$. Since A is G_m-closed, $c_m(A) \subset U$. Since A is s-closed, by

Lemma 3.1.12, $c_m(A) = c_m(c_\mu(A)) = c_\mu(c_m(A))$. That $c_\mu(c_m(A)) \subset U$ and $c_m(c_\mu(A)) \subset U$. Hence A is G-closed.

Definition 3.2.11 Let (X, μ, m) be a GTMS space. A subset A of X is said to be a G_m-open set if $X \setminus A$ is G_m-closed.

Theorem 3.2.12 Let (X, μ, m) be a GTMS space and $A \subset X$. Then A is G_m-open if and only if $F \subset i_m(A)$ whenever F is closed and $F \subset A$.

Proof. ($\rightarrow$) Assume that A is G_m-open. Let F be closed such that $F \subset A$. Then $X \setminus F$ is open and $X \setminus A \subset X \setminus F$. Since A is G_m-open, $X \setminus A$ is G_m-closed, and so $c_m(X \setminus A) \subset X \setminus F$. This implies $X \setminus i_m(A) \subset X \setminus F$. Hence $F \subset i_m(A)$.

($\leftarrow$) Assume that $F \subset i_m(A)$ whenever F is closed and $F \subset A$. To show that A is G_m-open, let U be open such that $X \setminus A \subset U$. Then $X \setminus U$ is closed and $X \setminus U \subset A$. By assumption, $X \setminus U \subset i_m(A)$. Thus $X \setminus i_m(A) \subset U$, and so $c_m(X \setminus A) \subset U$. Hence $X \setminus A$ is G_m-closed, and so A is G_m-open.

Proposition 3.2.13 Let (X, μ, m) be a GTMS space and $A \subset X$. If A is G_m-open and $i_m(A) \subset B \subset A$, then B is G_m-open.

Proof. Assume that A is G_m-open and $i_m(A) \subset B \subset A$. To show that B is G_m-open, let V be closed such that $V \subset B$. Since $V \subset B$ and $B \subset A$, $V \subset A$. From A is G_m-open and V is closed which $V \subset A$, By Theorem 3.2.12, $V \subset i_m(A)$. Since $i_m(A) \subset B$, $i_m(i_m(A)) \subset i_m(B)$, and so $i_m(A) \subset i_m(B)$. Hence $V \subset i_m(B)$. Then B is G_m-open.

3.3 Relation of G_μ-closed set and G_m-closed set

In this section, we will discuss some relations of G_μ-closed set and G_m-closed set and investigate some of their properties.

Theorem 3.3.1 Let (X,μ,m) be a GTMS space such that $X \notin \mu$ and $A \subset X$. If A is G_μ-closed, then A is G_m-closed.

Proof. Assume that A is G_μ-closed. Suppose to the contrary that A is not G_m-closed. Then there exists an open set U such that $A \subset U$ and $c_m(A) \not\subset U$. Since A is G_μ-closed, $c_\mu(A) \subset U$. From $c_\mu(A)$ is μ-closed, thus $X \setminus c_\mu(A)$ is μ-open. Since U is open, U is μ-open, and so $X \setminus c_\mu(A) \cup U$ is μ-open. Since $c_\mu(A) \subset U$, we have $X = (X \setminus c_\mu(A)) \cup c_\mu(A) \subset (X \setminus c_\mu(A)) \cup U \subset X$. Hence $(X \setminus c_\mu(A)) \cup U = X$. This implies $X \in \mu$. It is a contradiction with $X \notin \mu$. Thus A is G_m-closed.

Proposition 3.3.2 Let (X,μ,m) be a GTMS space such that $X \notin \mu$ and $A \subset X$. If A is G_μ-open, then A is G_m-open.

Proof. Assume that A is G_μ-open then $X \setminus A$ is G_μ-closed. By theorem 3.3.1 thus $X \setminus A$ is G_m-closed. Therefore A is G_m-open.

Proposition 3.3.3 Let (X,μ,m) be a GTMS space and $A \subset X$. Then A is G_μ-closed and $c_\mu(A)$ is G_m-closed if and only if A is $m\mu G$-closed.

Proof. (→) Assume that A is G_μ-closed and $c_\mu(A)$ is G_m-closed. To show that A is $m\mu G$-closed, let U be an open set such that $A \subset U$. Since A is G_μ-closed, $c_\mu(A) \subset U$. Since $c_\mu(A)$ is G_m-closed, $c_m(c_\mu(A)) \subset U$. Thus A is $m\mu G$-closed.

(←) Assume that A is $m\mu G$-closed. First, we will prove that A is G_μ-closed. Let U be an open set such that $A \subset U$. Since A is $m\mu G$-closed, $c_m(c_\mu(A)) \subset U$. But $c_\mu(A) \subset c_m(c_\mu(A))$, we obtain that $c_\mu(A) \subset U$. Hence A is G_μ-closed. Next, we will prove that $c_\mu(A)$ is G_m-closed. Let U be an open set such that $c_\mu(A) \subset U$. Then $A \subset U$. Since A is $m\mu G$-closed, $c_m(c_\mu(A)) \subset U$. Then $c_\mu(A)$ is G_m-closed.

Proposition 3.3.4 Let (X,μ,m) be a GTMS space and $A\subset X$. Then A is G_m-closed and $c_m(A)$ is G_μ-closed if and only if A is μmG-closed.

Proof. (1) (→) Assume that A is G_m-closed and $c_m(A)$ is G_μ-closed. To show that A is μmG-closed, let U be an open set such that $A\subset U$. Since A is G_m-closed, $c_m(A)\subset U$. Since $c_m(A)$ is G_μ-closed, $c_\mu(c_m(A))\subset U$. Thus A is μmG-closed.

(←) Assume that A is μmG-closed. First, we will prove that A is G_m-closed. Let U be an open set such that $A\subset U$. Since A is μmG-closed. $c_\mu(c_m(A))\subset U$. But $c_m(A)\subset c_\mu(c_m(A))$, we obtain that $c_m(A)\subset U$. Hence A is G_m-closed. Next, we will prove that $c_m(A)$ is G_μ-closed. Let U be an open set such that $c_m(A)\subset U$. Then $A\subset U$. Since A is μmG-closed, $c_\mu(c_m(A))\subset U$. Then $c_m(A)$ is G_μ-closed.

Proposition 3.3.5 Let (X,μ,m) be a GTMS space and A is s-closed in X. Then A is G_μ-closed if and only if A is G_m-closed.

Proof. (→) Assume that A is G_μ-closed. Let U be open such that $A\subset U$, that $c_\mu(A)\subset U$. Since A is s-closed, $c_\mu(A)=c_m(A)$, and so $c_m(A)\subset U$. Hence A is G_m-closed.

(←) Assume that A is G_m-closed. Let U be open such that $A\subset U$, that $c_m(A)\subset U$. Since A is s-closed, $c_\mu(A)=c_m(A)$, and so $c_\mu(A)\subset U$. Hence A is G_μ-closed.

CHAPTER 4

GT_1 -GTMS space and GT_2 - GTMS space

In this chapter, we will introduce the notions of GT_1-GTMS spaces and GT_2-GTMS spaces.

4.1 GT_1-GTMS space

Definition 4.1.1 A GTMS space (X,μ,m) is GT_1-GTMS space if and only if for pair of distinct point x and y in X, there exist a G_μ-open set U and a G_m-open set V such that $x\in U, y\notin U$, and $y\in V$, $x\notin V$.

Example 4.1.2 Let $X=\{1,2,3,4\}$ with generalized topology $\mu=\{\phi,\{1,2\},\{1,3\},\{2,3\},\{1,2,3\}\}$ and minimal structure $m=\{\phi,\{1\},\{1,2\},\{1,3\},\{2,3\},X\}$. Then $\phi,\{1,2\},\{1,3\},\{2,3\}$ are open. Hence $\phi,\{1\},\{2\},\{3\},\{4\},\{1,2\},\{1,3\},\{2,3\},\{1,2,3\}$ are G_μ-open and $\phi,\{1\},\{2\},\{3\},\{4\}$, $\{1,2\},\{1,3\},\{2,3\},\{1,2,3\},X$ are G_m-open. Then (X,μ,m) is GT_1-GTMS.

Example 4.1.3 Let $X=\{1,2,3\}$ with generalized topology $\mu=\{\phi,\{1\},\{1,2\},X\}$ and minimal structure $m=\{\phi,\{3,\},\{1,2\},X\}$. Then $\phi,\{1,2\},X$ are open. Hence $\phi,\{1\},\{2\},\{1,2\},X$ are G_μ-open and $\phi,\{1\},\{2\},\{3\},\{1,2\},\{1,3\}$, $\{2,3\},X$ are G_m-open. Then (X,μ,m) is not GT_1-GTMS.

Proposition 4.1.4 If (X,μ,m) is T_1-GTMS, then (X,μ,m) is GT_1-GTMS.

Proof. Assume that (X,μ,m) is T_1-GTMS. To show that (X,μ,m) is GT_1-GTMS, let $x,y\in X$ be such that $x\neq y$. Since X is T_1-GTMS, there exist μ-open U and m-

open V such that $x \in U, y \notin U$, and $y \in V$, $x \notin V$. By Proposition 3.1.3, 3.2.3, U is G_μ-open and V is G_m-open. Therefore (X, μ, m) is T_1-GTMS.

Theorem 4.1.5 Let (X, μ, m) be a GTMS space such that X has at least two elements and $X \notin \mu$. Then X is GT_1-GTMS if and only if $\{a\}$ is G_μ-open in X for all $a \in X$.

Proof. ($\rightarrow$) Assume that X is GT_1-GTMS. To show that $\{a\}$ is G_μ-open in X for all $a \in X$, let $a \in X$. Suppose $\{a\}$ is not G_μ-open. Then there exists a closed set F such that $F \subset \{a\}$ and $F \not\subset i_\mu(\{a\})$. This implies $\{a\} = F$ is closed. Thus $X \setminus \{a\}$ is open. Since X has a least two elements, $X \setminus \{a\} \neq \phi$, say $b \in X \setminus \{a\}$.

By assumption, there exist a G_μ-open set U and a G_m-open set V such that $a \in U, b \notin U$ and $b \in V, a \notin V$. Since U is G_μ-open and $\{a\}$ is closed such that $\{a\} \subset U$, $\{a\} \subset i_\mu(U)$. Then $X = (X \setminus \{a\}) \cup i_\mu(U) \in \mu$ which contradicts with $X \notin \mu$. Hence $\{a\}$ is G_μ-open.

($\leftarrow$) Assume that $\{a\}$ is G_μ-open in X for all $a \in X$. To show X is GT_1-GTMS, let $x, y \in X$ be such that $x \neq y$. By assumption, $\{x\}$ and $\{y\}$ is G_μ-open. Since $X \notin \mu$, by Proposition 3.3.2, $\{y\}$ is G_m-open. Set $U = \{x\}$ and $\{y\} = V$. Then U is G_μ-open and V is G_m-open. Moreover, $x \in U, y \notin U$ and $y \in V, x \notin V$. Hence X is GT_1-GTMS.

Definition 4.1.6 Let (X, μ, m) be a GTMS space and $A \subset X$. Then

1. $c_{G_\mu}(A) = \cap\{K : K$ is G_μ-closed and $A \subset K\}$,
2. $c_{G_m}(A) = \cap\{K : K$ is G_m-closed and $A \subset K\}$.

Lemma 4.1.7 Let (X, μ, m) be a GTMS space and $A \subset X$. Then $x \in c_{G_\mu}(A)$ if and only if $A \cap U \neq \phi$ for all G_μ-open U containing x.

Proof. (→) Assume that there exists a G_μ-open set U containing x such that $A \cap U = \phi$. Then $X \setminus U$ is G_μ-closed and $A \subset X \setminus U$. Since $x \notin X \setminus U$, $x \notin c_{G_\mu}(A)$.

(←) Assume that $x \notin c_{G_\mu}(A)$. Then there exists a G_μ-closed set K such that $A \subset K$ and $x \notin K$. Thus $X \setminus K$ is G_μ-open and $x \in X \setminus K$. Moreover $A \cap (X \setminus K) = \phi$.

Lemma 4.1.8 Let (X,μ,m) be a GTMS space and $A \subset X$. Then $x \in c_{G_m}(A)$ if and only if $A \cap U \neq \phi$ for all G_m-open U containing x.

Proof. (→) Assume that there exists a G_m-open set U containing x such that a $A \cap U = \phi$. Then $X \setminus U$ is G_m-closed and $A \subset X \setminus U$. Since $x \notin X \setminus U$, $x \notin c_{G_m}(A)$.

(←) Assume that $x \notin c_{G_m}(A)$. Then there exists a G_m-closed set K such that $A \subset K$ and $x \notin K$, Thus $X \setminus K$ is G_m-open and $x \in X \setminus K$. Moreover $A \cap (X \setminus K) = \phi$.

Theorem 4.1.9 Let (X,μ,m) be a GTMS space and $A \subset X$. Then the following are equivalent:

1. (X,μ,m) is GT_1-GTMS.
2. $c_{G_\mu}(\{x\}) = \{x\}$ and $c_{G_m}(\{x\}) = \{x\}$.
3. $c_{G_\mu}(c_{G_m}(\{x\})) = \{x\}$.
4. $c_{G_m}(c_{G_\mu}(\{x\})) = \{x\}$.

Proof. (1⟶2) Assume that X is GT_1-GTMS. To show that $c_{G_\mu}(\{x\}) = \{x\}$, and $c_{G_m}(\{x\}) = \{x\}$ for all $x \in X$, let $x \in X$. It is clear that $\{x\} \subset c_{G_\mu}(\{x\})$. Let $y \in X$ be such that $y \neq x$. By assumption, there exists a G_μ-open set U such that $y \in U$ but $x \notin U$. Then $U \cap \{x\} = \phi$. Thus $y \notin c_{G_\mu}(\{x\})$. Hence $c_{G_\mu}(\{x\}) \subset \{x\}$. Then $c_{G_\mu}(\{x\}) = \{x\}$. Similarly, we can prove $c_{G_m}(\{x\}) = \{x\}$ for all $x \in X$.

(2⟶1) Assume that $c_{G_\mu}(\{x\})=\{x\}$ and $c_{G_m}(\{x\})=\{x\}$. To show that X is GT_1-GTMS, let $x,y\in X$ with $x\neq y$. By assumption that is $c_{G_m}(\{x\})=\{x\}$ and $c_{G_\mu}(\{y\})=\{y\}$. Thus $x\notin c_{G_\mu}(\{y\})$ and $y\notin c_{G_m}(\{x\})$. Since $x\notin c_{G_\mu}(\{y\})$, that G_μ-open. Then there exist a G_μ-open set U and G_m-open set V such that $x\in U$, $\{y\}\cap U=\phi$ and $y\in V$, $\{x\}\cap V=\phi$. Hence X is GT_1-GTMS .

It is clear that (2⟷3) and (2⟷4).

Definition 4.1.10 A GTMS space (X,μ,m) is called a GT_0-GTMS space if and only if for any pair of distinct point x and y in X, there exists a subset U of X which is G_μ-open or G_m-open such that $x\in U, y\notin U$ or $y\in U, x\notin U$.

Definition 4.1.11 A GTMS space (X,μ,m) is a GR_0-GTMS space if and only if for each $x,y\in X$ if $x\in c_{G_\mu}(c_{G_m}(\{y\}))$, then $y\in c_{G_\mu}(\{x\})$ and if $x\in c_{G_m}(c_{G_\mu}(\{y\}))$, then $y\in c_{G_m}(\{x\})$.

Definition 4.1.12 A GTMS space (X,μ,m) is a G-symmetric GTMS space if and only if for each $x,y\in X$ if $y\in c_{G_\mu}(c_{G_m}(\{x\}))$, then $x\in c_{G_\mu}(c_{G_m}(\{y\}))$ and if $y\in c_{G_m}(c_{G_\mu}(\{x\}))$, then $x\in c_{G_m}(c_{G_\mu}(\{y\}))$.

Proposition 4.1.13 If (X,μ,m) is GT_1-GTMS, then (X,μ,m) is GT_0-GTMS.

Proof. Assume that (X,μ,m) is GT_1-GTMS. To show that (X,μ,m) is GT_0-GTMS, let $x,y\in X$ with $x\neq y$. Since (X,μ,m) is GT_1-GTMS, there exist a G_μ-open set U and a G_m-open set V such that $x\in U, y\notin U$ and $y\in V, x\notin V$. Hence (X,μ,m) is GT_0-GTMS.

Remark The converse of the previous proposition need not be true as the following example.

Example 4.1.14 Let $X=\{1,2,3\}$ with generalized topology $\mu=\{\phi,\{3\},\{2,3\},X\}$ and minimal structure $m=\{\phi,\{3\},\{1,2\},X\}$.Then $\phi,\{1,2\},X$ are open. Hence $\phi,\{1\},\{2\}$, $\{1,2\},X$ are G_μ-open and $\phi,\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\},X$ are G_m-open. (X,μ,m) is GT_0-GTMS, but (X,μ,m) is not GT_1-GTMS.

Proposition 4.1.15 If (X,μ,m) is GT_1-GTMS, then (X,μ,m) is GR_0-GTMS.

Proof. Assume that (X,μ,m) is GT_1-GTMS. To show that (X,μ,m) is GR_0-GTMS, let $x,y\in X$ with $x\in c_{G_\mu}(c_{G_m}(\{y\}))$. Since X is GT_1-GTMS, $c_{G_\mu}(c_{G_m}(\{y\}))=\{y\}$. Then $x\in\{y\}$, and so $x=y$. Hence $y\in c_{G_\mu}(\{x\})$. Similarly, we can prove that if $x\in c_{G_m}(c_{G_\mu}(\{y\}))$, then $y\in c_{G_m}(\{x\})$.

Proposition 4.1.16 If (X,μ,m) is GT_1-GTMS, then (X,μ,m) is G-symmetric GTMS.

Proof. Assume that (X,μ,m) is GT_1-GTMS. To show that (X,μ,m) is G-symmetric GTMS, let $x,y\in X$ with $y\in c_{G_\mu}(c_{G_m}(\{x\}))$. Since X is GT_1-GTMS, $c_{G_\mu}(c_{G_m}(\{x\}))=\{x\}$. Then $y\in\{x\}$, and so $y=x$. Hence $x\in c_{G_\mu}(c_{G_m}(\{y\}))$. Similarly, we can prove that if $y\in c_{G_m}(c_{G_\mu}(\{x\}))$, then $x\in c_{G_m}(c_{G_\mu}(\{y\}))$.

Theorem 4.1.17 Let (X,μ,m) be a GTMS space. Then (X,μ,m) is GT_1-GTMS if and only if (X,μ,m) is GT_0-GTMS, GR_0-GTMS and G-symmetric GTMS space.

Proof. (→) It follows from Proposition 4.1.13, 4.1.15, 4.1.16.

(←) Assume that (X,μ,m) is GT_0-GTMS, GR_0-GTMS and G-symmetric GTMS. To show that (X,μ,m) is GT_1-GTMS, let $x,y\in X$ with $y\neq x$. Since X is GT_0-GTMS, there exists a subset U of X which is G_μ-open or G_m-open such that $x\in U, y\notin U$ or $y\in U, x\notin U$.

Case 1 U is G_μ-open

Subcase 1.1 $x\in U, y\notin U$

Thus $x \notin c_{G_\mu}(\{y\})$. Since X is GR_0-GTMS, thus $x \notin c_{G_\mu}(c_{G_m}(\{y\}))$. Since X is G-symmetric GTMS, $y \notin c_{G_\mu}(c_{G_m}(\{x\}))$. Hence $c_{G_\mu}(c_{G_m}(\{x\})) \subset \{x\}$. Therefore, $c_{G_\mu}(c_{G_m}(\{x\})) = \{x\}$.

Subcase 1.2 $y \in U, x \notin U$

Thus $y \notin c_{G_\mu}(\{x\})$. Since X is GR_0-GTMS, thus $y \notin c_{G_\mu}(c_{G_m}(\{x\}))$. Hence $c_{G_\mu}(c_{G_m}(\{x\})) \subset \{x\}$. Therefore, $c_{G_\mu}(c_{G_m}(\{x\})) = \{x\}$.

Case 2 U is G_m-open

Subcase 2.1 $x \in U, y \notin U$

Thus $x \notin c_{G_m}(\{y\})$. Since X is GR_0-GTMS, thus $x \notin c_{G_m}(c_{G_\mu}(\{y\}))$. Since X is G-symmetric GTMS, $y \notin c_{G_m}(c_{G_\mu}(\{x\}))$. Hence $c_{G_m}(c_{G_\mu}(\{x\})) \subset \{x\}$. Therefore, $c_{G_m}(c_{G_\mu}(\{x\})) = \{x\}$.

Subcase 2.2 $y \in U, x \notin U$

Thus $y \notin c_{G_m}(\{x\})$. Since X is GR_0-GTMS, $y \notin c_{G_m}(c_{G_\mu}(\{x\}))$. Hence $c_{G_m}(c_{G_\mu}(\{x\})) \subset \{x\}$. Therefore, $c_{G_m}(c_{G_\mu}(\{x\})) \subset \{x\}$. Hence X is GT_1-GTMS.

4.2 GT_2-GTMS space

Definition 4.2.1 A GTMS space (X, μ, m) is GT_2-GTMS if and only if for any pair of distinct point x and y in X, there exist a G_μ-open set U and a G_m-open set V such that $x \in U$, $y \in V$ and $U \cap V = \phi$.

Example 4.2.2 Let $X = \{1,2,3,4\}$ with generalized topology
$\mu = \{\phi, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$ and minimal structure
$m = \{\phi, \{1\}, \{1,2\}, \{1,3\}, \{2,3\}, X\}$. Then $\phi, \{1,2\}, \{1,3\}, \{2,3\}$ are open. Hence $\phi, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}$ are G_μ-open and $\phi, \{1\}, \{2\}, \{3\}, \{4\}$, $\{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}, X$ are G_μ-open. Then (X, μ, m) is GT_2-GTMS.

Example 4.2.3 Let $X=\{1,2,3\}$ with generalized topology $\mu=\{\phi,\{1,2\},\{1,3\},\{2,3\},X\}$ and minimal structure $m=\{\phi,\{1,2\},\{1,3\},\{2,3\},X\}$. Then $\phi,\{1,2\},\{1,3\},\{2,3\},X$ are open. Hence $\phi,\{1,2\},\{1,3\},\{2,3\},X$ are G_μ-open and $\phi,\{1,2\},\{1,3\},\{2,3\},X$ are G_m-open. Then (X,μ,m) is not GT_2-GTMS.

Proposition 4.2.4 If (X,μ,m) is T_2-GTMS, then (X,μ,m) is GT_2-GTMS.

Proof. Assume that (X,μ,m) is T_2-GTMS. To show that (X,μ,m) is GT_2-GTMS, let $x,y\in X$ be such that $x\neq y$. Since X is T_2-GTMS, there exist disjoint μ-open U and m-open V such that $x\in U$, $y\in V$ and $U\cap V=\phi$. By Proposition 3.1.3, 3.2.3, U is G_μ-open and V is G_m-open. Therefore (X,μ,m) is GT_1-GTMS.

Theorem 4.2.5 Let (X,μ,m) be a GTMS space and $A\subset X$. Then the following are equivalent.

1. X is a GT_2-GTMS space.
2. If $x\in X$, then for each $y\neq x$, there exists a G_μ-open set U containing x such that $y\notin c_{G_m}(U)$.
3. For each $x\in X, \{x\}=\cap\{c_{G_m}(U):U$ is G_μ-open and $x\in U\}$.

Proof. (1→2) Assume that X is GT_2-GTMS and $x\in X$. Let $y\in X$ with $y\neq x$. There exists a G_μ-open set U and a G_m-open set V such that $x\in U, y\in V$ and $U\cap V=\phi$. Hence $y\notin c_{G_m}(U)$.

(2→3) Let $x\in X$. We will prove that $\{x\}=\cap\{c_{G_m}(U):U$ is G_μ-open and $x\in U\}$. Let $y\in X$ with $y\neq x$. By assumption there exists G_μ-open set U containing x such that $y\notin c_{G_m}(U)$. Thus $y\notin\cap\{c_{G_m}(U):U$ is G_μ-open and $x\in U\}$. Then $\cap\{c_{G_m}(U):U$ is G_μ-open and $x\in U\}\subseteq\{x\}$. Hence $\{x\}=\cap\{c_{G_m}(U):U$ is G_μ-open and $x\in U\}$.

(3→1) Assume that $\{x\}=\cap\{c_{G_m}(U):U$ is G_μ-open and $x\in U\}$ for each $x\in X$. Let $x,y\in X$ which $y\neq x$. Since $y\notin\{x\}=\cap\{c_{G_m}(U):U$ is G_μ-open and $x\in U\}$. There a G_μ-open set U_1 such that $x\in U_1$, $y\notin c_{G_m}(U_1)$. Since $y\notin c_{G_m}(U_1)$, there is a G_m-open set V_1 containing y such that $U_1\cap V_1=\phi$. Hence $x\in U_1, y\in V_1$ and $U_1\cap V_1=\phi$. Therefore X is GT_2-GTMS.

Proposition 4.2.6 If (X,μ,m) is GT_2-GTMS, then (X,μ,m) is GT_1-GTMS.

Proof. Assume that (X,μ,m) is GT_2-GTMS. To show that (X,μ,m) is GT_1-GTMS, let $x,y\in X$ be such that $x\neq y$. Since X is GT_2-GTMS, there exist G_μ-open U and G_m-open V such that $x\in U, y\in V$ and $U\cap V=\phi$. Then $x\in U, y\notin U$ and $y\in V, x\notin V$. Hence X is GT_1-GTMS.

Remark The converse of the previous proposition need not be true as the following example.

Example 4.2.7 Let $X=\{1,2,3\}$ with generalized topology

$\mu=\{\phi,\{1,2\},\{1,3\},\{2,3\},X\}$ and minimal structure $m=\{\phi,\{1,2\},\{1,3\},\{2,3\},X\}$.

Then $\phi,\{1,2\},\{1,3\},\{2,3\},X$ are open. Hence $\phi,\{1,2\},\{1,3\},\{2,3\},X$ are G_μ-open and $\phi,\{1,2\},\{1,3\},\{2,3\},X$ are G_m-open. Thus (X,μ,m) is GT_1-GTMS, but (X,μ,m) is not GT_2-GTMS.

Definition 4.2.8 A GTMS space is a GR_1-GTMS if and only if for all $x,y\in X$ with $x\neq y$ if $c_{G_\mu}(\{x\})\neq c_{G_m}(\{y\})$, then there exist disjoint G_m-open U and G_μ-open V such that $c_{G_\mu}(\{x\})\subset V$, $c_{G_m}(\{y\})\subset U$.

Proposition 4.2.9 If (X,μ,m) is GR_1-GTMS, then (X,μ,m) is GR_0-GTMS.

Proof. Assume that (X,μ,m) is GR_1-GTMS. To show that (X,μ,m) is GR_0-GTMS, let $x,y\in X$ with $y\notin c_{G_\mu}(\{x\})$. Since $x\neq y$, thus $c_{G_\mu}(\{x\})\neq c_{G_m}(\{y\})$. Then there exist a G_μ-open set V and a G_m-open set U such that $c_{G_\mu}(\{x\})\subset V$ and $c_{G_m}(\{y\})\subset U$. Hence $c_{G_m}(\{y\})\cap V=\phi$. Therefore $x\notin c_{G_\mu}(c_{G_m}(\{y\}))$. Similarly, we can prove that if $y\notin c_{G_m}(\{x\})$, then $x\notin c_{G_m}(c_{G_\mu}(\{y\}))$.

Proposition 4.2.10 If (X,μ,m) is GR_1-GTMS, then (X,μ,m) is G-symmetric GTMS.

Proof. Assume that (X,μ,m) is GR_1-GTMS. To show that (X,μ,m) is G-symmetric GTMS, let $x,y\in X$ with $x\notin c_{G_m}(c_{G_\mu}(\{y\}))$. Thus $x\notin c_{G_m}(\{y\})$. Hence $x\neq y$ and $c_{G_\mu}(\{x\})\neq c_{G_m}(\{y\})$. Then there exist a G_μ-open set U and a G_m-open set V such that $c_{G_\mu}(\{x\})\subset U$ and $c_{G_m}(\{y\})\subset V$. Hence $c_{G_\mu}(\{x\})\cap V=\phi$. Therefore $y\notin c_{G_m}(c_{G_\mu}(\{x\}))$. Similarly, we can prove that if $x\notin c_{G_\mu}(c_{G_m}(\{y\}))$, then $y\notin c_{G_\mu}(c_{G_m}(\{x\}))$

Proposition 4.2.11 Let (X,μ,m) be a GTMS space. Then (X,μ,m) is GR_1-GTMS and GT_1-GTMS if and only if (X,μ,m) is GT_2-GTMS.

Proof. ($\rightarrow$) Assume that (X,μ,m) is GR_1-GTMS and GT_1-GTMS. To show that (X,μ,m) is GT_2-GTMS, let $x,y\in X$ with $x\neq y$. Since X is GT_1-GTMS, thus $c_{G_\mu}(\{x\})=\{x\}=c_{G_m}(\{x\})$ and $c_{G_\mu}(\{y\})=\{y\}=c_{G_m}(\{y\})$. Thus $c_{G_\mu}(\{x\})\neq c_{G_m}(\{y\})$. Since X is GR_1-GTMS, then there exist disjoint G_m-open U and G_μ-open V such that $c_{G_\mu}(\{x\})\subset V$ and $c_{G_m}(\{y\})\subset U$. Therefore X is GT_2-GTMS.

($\leftarrow$) Assume that (X,μ,m) is GT_2-GTMS. By Proposition 4.2.6, (X,μ,m) is GT_1-GTMS. Next, we will show that (X,μ,m) is GR_1-GTMS. Let $x,y\in X$ with $c_{G_\mu}(\{x\})\neq c_{G_m}(\{y\})$. Since X is GT_1-GTMS, $c_{G_\mu}(\{x\})=\{x\}=c_{G_m}(\{x\})$ and $c_{G_\mu}(\{y\})=\{y\}=c_{G_m}(\{y\})$. This implies $x\neq y$. Since X is GT_2-GTMS, then there exist disjoint G_m-open U and G_μ-open V such that $c_{G_\mu}(\{x\})=\{x\}\subset V$ and $c_{G_m}(\{y\})=\{y\}\subset U$. Therefore X is GR_1-GTMS.

CHAPTER 5

CONCLUSIONS AND RECOMMENDATIONS

5.1 Conclusions

The aim of this thesis is to study the G_μ-closed set, G_m-closed set and introduce some separation axioms in GTMS space using G_μ-open and G_m-open. Moreover, we introduce GT_1-GTMS space and GT_2-GTMS space and study some of their properties. The results are follows:

1) Let (X,μ,m) be a GTMS space. A subset A of X is said to be a G_μ-closed set if $c_\mu(A)\subset U$ whenever $A\subset U$ and U is open.

From the above definition, I the following theorems are derived:

1.1) Let (X,μ,m) be a GTMS space and $A\subset X$. If A is μ-closed, then A is G_μ-closed.

1.2) Every μmG-closed set in a GTMS space (X,μ,m) is G_μ-closed.

1.3) Every $m\mu G$-closed set in a GTMS space (X,μ,m) is G_μ-closed.

1.4) Let (X,μ,m) be a GTMS space and $A,F\subset X$. If A is G_μ-closed and F is closed, then $A\cap F$ is G_μ-closed.

1.5) Let (X,μ,m) be a GTMS space and $A\subset X$. If A is open and G_μ-closed, then A is μ-closed.

1.6) Let (X,μ,m) be a GTMS space and $A\subset X$. If A is G_μ-closed, then $c_\mu(A)\setminus A$ does not contain any nonempty closed set.

1.7) Let (X,μ,m) be a GTMS space and $A\subset X$. If A is s-closed, then $c_\mu(c_m(A))=c_\mu(A)=c_m(A)=c_m(c_\mu(A))$.

1.8) Let (X,μ,m) be a GTMS space and $A\subset X$. If A is G_μ-closed and s-closed, then A is G-closed.

2) Let (X,μ,m) be a GTMS space. A subset A of X is said to be a G_μ-open set if $X\setminus A$ is G_μ-closed.

From the above definition, I the following theorems are derived:

2.1) Let (X,μ,m) be a GTMS space and $A\subset X$. Then A is G_μ-open if and only if $F\subset i_\mu(A)$ whenever F is closed and $F\subset A$.

2.2) Let (X,μ,m) be a GTMS space and $A\subset X$. If A is G_μ-closed, then $c_\mu(A)\setminus A$ is G_μ-open.

2.3) Let (X,μ,m) be a GTMS space and $A\subset X$. If A is G_μ-open and $i_\mu(A)\subset B\subset A$, then B is G_μ-open.

2.4) Let (X,μ,m) be a GTMS space and $A\subset X$. If A is G_μ-open, then $U=X$ whenever U is open and $i_\mu(A)\cup(X\setminus A)\subset U$.

2.5) Let (X,μ,m) be a GTMS space such that μ is a QT on X. If A and B are G_μ-closed, then $A\cup B$ is G_μ-closed.

2.6) Let (X,μ,m) be a GTMS space with $X\notin\mu$ and $A,B\subset X$. If A is G_μ-closed $A\subset B$, then B is G_μ-closed.

2.7) Let (X,μ,m) be a GTMS space with $X\notin\mu$, if A or B are G_μ-closed, then $A\cup B$ is G_μ-closed.

3) Let (X,μ,m) be a GTMS space. A subset A of X is said to be a G_m-closed set if $c_m(A)\subset U$ whenever $A\subset U$ and U is open.

From the above definition, I the following theorems are derived:

3.1) Let (X,μ,m) be a GTMS space and $A\subset X$. If A is m-closed, then A is G_m-closed.

3.2) Every μmG-closed set in a GTMS space (X,μ,m) is G_m-closed.

3.3) Every $m\mu G$-closed set in a GTMS space (X,μ,m) is G_m-closed.

3.4) Let (X, μ, m) be a GTMS space and $A, F \subset X$. If A is G_m-closed and F is closed, then $A \cap F$ is G_m-closed.

3.5) Let (X, μ, m) be a GTMS space and $A \subset X$. If A is open and G_m-closed and m has the property B, then A is m-closed.

3.6) Let (X, μ, m) be a GTMS space and $A \subset X$. If A is G_m-closed and s-closed, then A is G-closed.

4) Let (X, μ, m) be a GTMS space. A subset A of X is said to be a G_m-open set if $X \setminus A$ is G_m-closed.

From the above definition, I the following theorems are derived:

4.1) Let (X, μ, m) be a GTMS space and $A \subset X$. Then A is G_m-open if and only if $F \subset i_m(A)$ whenever F is closed and $F \subset A$.

4.2) Let (X, μ, m) be a GTMS space and $A \subset X$. If A is G_m-open and $i_m(A) \subset B \subset A$, then B is G_m-open.

4.3) Let (X, μ, m) be a GTMS space such that $X \notin \mu$ and $A \subset X$. If A is G_μ-closed, then A is G_m-closed.

4.4) Let (X, μ, m) be a GTMS space such that $X \notin \mu$ and $A \subset X$. If A is G_μ-open, then A is G_m-open.

4.5) Let (X, μ, m) be a GTMS space and $A \subset X$. Then A is G_μ-closed and $c_\mu(A)$ is G_m-closed if and only if A is $m\mu G$-closed.

4.6) Let (X, μ, m) be a GTMS space and $A \subset X$. Then A is G_m-closed and $c_m(A)$ is G_μ-closed if and only if A is $\mu m G$-closed.

4.7) Let (X, μ, m) be a GTMS space and A is s-closed in X. Then A is G_μ-closed if and only if A is G_m-closed.

5) A GTMS space (X,μ,m) is GT_1-GTMS space if and only if for pair of distinct point x and y in X, there exist a G_μ-open set U and a G_m-open set V such that $x \in U, y \notin U$, and $y \in V$, $x \notin V$.

From the above definition, I the following theorems are derived:

5.1) If (X,μ,m) is T_1-GTMS, then (X,μ,m) is GT_1-GTMS.

5.2) Let (X,μ,m) be a GTMS space such that X has a least two elements and $X \notin \mu$. Then X is GT_1-GTMS if and only if $\{a\}$ is G_μ-open in X for all $a \in X$.

6) Let (X,μ,m) be a GTMS space and $A \subset X$. Then

1. $c_{G_\mu}(A) = \cap\{K : K$ is G_μ-closed and $A \subset K\}$,

2. $c_{G_m}(A) = \cap\{K : K$ is G_m-closed and $A \subset K\}$.

From the above definition, I the following theorems are derived:

6.1) Let (X,μ,m) be a GTMS space and $A \subset X$. Then $x \in c_{G_\mu}(A)$ if and only if $A \cap U \neq \phi$ for all G_μ-open U containing x.

6.2) Let (X,μ,m) be a GTMS space and $A \subset X$. Then $x \in c_{G_m}(A)$ if and only if $A \cap U \neq \phi$ for all G_m-open U containing x.

6.3) Let (X,μ,m) be a GTMS space and $A \subset X$. Then the following are equivalent:

1. (X,μ,m) is GT_1-GTMS.
2. $c_{G_\mu}(\{x\}) = \{x\}$ and $c_{G_m}(\{x\}) = \{x\}$.
3. $c_{G_\mu}(c_{G_m}(\{x\})) = \{x\}$.
4. $c_{G_m}(c_{G_\mu}(\{x\})) = \{x\}$.

7) A GTMS space (X,μ,m) is called a GT_0-GTMS space if and only if for any pair of distinct point x and y in X, there exists a subset U of X which is G_μ-open or G_m-open such that $x \in U, y \notin U$ or $y \in U, x \notin U$.

8) A GTMS space (X,μ,m) is a GR_0-GTMS space if and only if for each $x,y\in X$ if $x\in c_{G_\mu}(c_{G_m}(\{y\}))$, then $y\in c_{G_\mu}(\{x\})$ and if $x\in c_{G_m}(c_{G_\mu}(\{y\}))$, then $y\in c_{G_m}(\{x\})$.

9) A GTMS space (X,μ,m) is a G-symmetric GTMS space if and only if for each $x,y\in X$ if $y\in c_{G_\mu}(c_{G_m}(\{x\}))$, then $x\in c_{G_\mu}(c_{G_m}(\{y\}))$ and if $y\in c_{G_m}(c_{G_\mu}(\{x\}))$, then $x\in c_{G_m}(c_{G_\mu}(\{y\}))$.

From the above definition, I the following theorems are derived:

9.1) If (X,μ,m) is GT_1-GTMS, then (X,μ,m) is GT_0-GTMS.

9.2) If (X,μ,m) is GT_1-GTMS, then (X,μ,m) is GR_0-GTMS.

9.3) If (X,μ,m) is GT_1-GTMS, then (X,μ,m) is G-symmetric GTMS.

9.4) Let (X,μ,m) be a GTMS space. Then (X,μ,m) is GT_1-GTMS if and only if (X,μ,m) is GT_0-GTMS, GR_0-GTMS and G-symmetric GTMS space.

10) A GTMS space (X,μ,m) is GT_2-GTMS if and only if for any pair of distinct point x and y in X, there exist a G_μ-open set U and a G_m-open set V such that $x\in U$, $y\in V$ and $U\cap V=\phi$.

From the above definition, I the following theorems are derived:

10.1) If (X,μ,m) is T_2-GTMS, then (X,μ,m) is GT_2-GTMS.

10.2) Let (X,μ,m) be a GTMS space and $A\subset X$. Then the following are equivalent.

1. X is a GT_2-GTMS space.

2. If $x\in X$, then for each $y\neq x$, there exists a G_μ-open set U containing x such that $y\notin c_{G_m}(U)$.

3. For each $x\in X, \{x\}=\cap\{c_{G_m}(U): U$ is G_μ-open and $x\in U\}$.

10.3) If (X,μ,m) is GT_2-GTMS, then (X,μ,m) is GT_1-GTMS.

11) A GTMS space is a GR_1-GTMS if and only if for all $x, y \in X$ with $x \neq y$ if $c_{G_\mu}(\{x\}) \neq c_{G_m}(\{y\})$, then there exist disjoint G_m-open U and G_μ-open V such that $c_{G_\mu}(\{x\}) \subset V$, $c_{G_m}(\{y\}) \subset U$.

From the above definition, I the following theorems are derived:

11.1) If (X, μ, m) is GR_1-GTMS, then (X, μ, m) is GR_0-GTMS.

11.2) If (X, μ, m) is GR_1-GTMS, then (X, μ, m) is G-symmetric GTMS.

11.3) Let (X, μ, m) be a GTMS space. Then (X, μ, m) is GR_1-GTMS and GT_1-GTMS if and only if (X, μ, m) is GT_2-GTMS.

5.2 Recommendations

Even though, I have found several properties as of the sets and space presented in this thesis, there are several questions yet to be answered and it may be worth investigating in future studies. I formulate the questions as follows :

5.2.1 Are there another properties of G_μ-closed sets and G_m-closed sets in a GTMS space ?

5.2.2 Are there another properties of GT_1-GTMS space and GT_2-GTMS space ?

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