# Weakly Generalized Closed Sets in Ideal Topological Spaces

Walailuk Peanchai

A Thesis Submitted in Partial Fulfillment of Requirements for

degree of Master of Science in Mathematics Education

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A Thesis Submitted in Partial Fulfillment of Requirements

for Master of Science (Mathematics Education)

March 2020

The examining committee has unanimously approved this Thesis, submitted by Miss Walailuk Peanchai , as a partial fulfillment of the requirements for the Master of Science Mathematics Education at Mahasarakham University

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## ABSTRACT

In this research, we introduce and study the concept of weakly generalized closed sets in ideal topological spaces. Moreover, we study some separation axioms in ideal topological spaces. Furthermore, some properties of weakly generalized closed sets in ideal bitopological spaces are investigated.


Keyword : weakly generalized closed sets with respect to an ideal, weakly generalized open sets with respect to an ideal

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# CHAPTER 1

# INTRODUCTION

## 1.1 Background

General topology is important in many fields of applied science as well as branches of mathematics. In reality it is used in data mining, computational topology for geometric design and molecular design, computer-aided design, computer-aided geometric design, digital topology, information systems, particle physics and quantum physics etc. As is noticed from the recent literature, there has been a growing trend among some topologists to introduce and study generalized types of closed sets. Generalized closed sets and generalized open set, as significant and fundamental subjects in the study topology, have been researched by many mathematicians. In 1970, Levin [10] introduce the concept of generalized closed sets in a topological space by comparing the closure of a subset with its open supersets.

The study of generalized closed sets has produced some new separation axioms which lie been $T_0$ and $T_1$ such as $T_{\frac{1}{2}}$, $T_{gs}$ and $T_{\frac{3}{4}}$. Some of these properties have been found to be useful in computer science and digital topology [7]. Other new properties are define by variations of the property of submaximality. Furthermore, the study of generalized closed sets also provides new characterizations of some known classes of spaces, for example, the class of extremally disconnected spaces. As the weak form generalized closed sets, the notion of weakly generalized closed sets was introduced and studied by Sundaram and Nagaveni [17]. Sandaram and Pushpalatha and [18] introduced and studied the notion of strongly generalized closed sets, which is implies by that of closed sets and implies that of generalized closed sets. Park and Park [14] introduced and studied mildly generalized closed sets, which is property placed between the classes of strongly generalized closed sets and weakly generalized closed sets.

The concept of ideal topological space was studied by Kuratowski [8] and Vaidyanathaswamy [19]. Jankovic and Hamlett [2] investigated further properties of ideal topological spaces. Noiri and Rajesh [13] introduce and studied the concept of generalized closed sets with respect to an ideal in bitopological spaces. In 2011, Jafari and Rajesh [5] introduced and investigated the concept of generalized closed sets with respect to an ideal, which is extension of the concept of generalized closed sets. According to the prior studies as mentioned above, I am interested in define and studying some properties of weakly generalized closed sets in ideal topological and weakly generalized closed set with respect to an ideal in bitopological spaces.

The thesis is divided into five chapters. The first chapter is formed by an introduction which contains some historical remarks concerning the research specialization. We also explain our motivations and outline the goals of the thesis here. In the second chapter, we give some definitions, notations and some known theorems that will be used in the later chapter. In the third chapter, we introduce and study weakly generalized closed sets in ideal topological space. Moreover, we investigate some characterizations of weakly generalized I-normal and weakly generalized I-regular spaces. In the fourth chapter, we introduce and study the concept of weakly generalized closed set with respect to an ideal in bitopological spaces. In the last chapter, we make conclusions of the obtained results of the research.

## 1.2 Objective of the research

1.2.1 Define and study the notion of weakly generalized closed sets with respect to ideal in topological spaces.

1.2.2 Define and study the notion of weakly generalized $\tau_1\tau_2$-closed sets with respect to ideal in bitopological spaces.

## 1.3 Research methodology

1.3.1 Study about generalized closed sets with respect to topological spaces.

1.3.2 Study about generalized closed sets with respect to bitopological spaces.

1.3.3 Study about weakly generalized closed sets in ideal bitopological spaces.

1.3.4 Write the research for publish in international journals.

1.3.5 Summary the research and prepare complete research report offer to Mahasarakham Universty.

## 1.4 Scope of the study

1.4.1 Define and study the notion of weakly generalized closed sets in ideal topological spaces.

1.4.2 Define and study the notion of weakly generalized closed sets in ideal bitopological spaces.

# CHAPTER 2

# PRELIMINARIES

This chapter, we introduce some basic knowledge, definitions, notations and dealing with some preliminaries and give some useful results that will be duplicated in later chapter.

## 2.1 Some properties of topological spaces

In this section, we introduce some basic knowledge, definitions, notations and known propositions of topological spaces.

**Definition 2.1.1** [11] Let $X$ be a non-empty set and $\tau$ a collection of subsets of $X$ such that :

(1) $\varnothing \in \tau$.

(2) $X \in \tau$.

(3) If $G_1, G_2, \ldots, G_n \in \tau$ then $G_1 \cap G_2 \cap \ldots \cap G_n \in \tau$.

(4) If for each $\alpha \in I, G_\alpha \in \tau$, then $\bigcup_{\alpha \in I} G_\alpha \in \tau$.

The pair $(X, \tau)$ is called a topological space. The set $X$ is called the underlying set and the collection $\tau$ is called the topology on the set $X$. The elements of $\tau$ are called open sets and the complements are called closed sets.

**Definition 2.1.2** [17] Let $(X, \tau)$ be a topological space. A subset $F$ of $X$ is said to be closed in $X$ if $X - F$ is open in $X$.

**Definition 2.1.3** [11] Let $X$ be a non-empty set and $\tau$ a topology on $X$. For a subset $A$ of $X$, the closure and the interior of $A$, denoted by $Cl(A)$ and $Int(A)$, respectively, are defined as follows :

(1) $Cl(A) = \cap\{F \mid A \subseteq F, X - F \in \tau\}$.

(2) $Int(A) = \cup\{G \mid G \subseteq A, G \in \tau\}$.

**Theorem 2.1.4** [3] Let $A$ and $B$ be subset of a topological space $(X, \tau)$. Then

(1) $A \subseteq Cl(A)$.

(2) If $A \subseteq B$, then $Cl(A) \subseteq Cl(B)$.

(3) $A$ is closed in $X$ if and only if $A = Cl(A)$.

(4) $Cl(A)$ is the smallest closed set in $X$ with $A \subseteq Cl(A)$.

(5) $Cl(Cl(A)) = Cl(A)$.

(6) $Cl(A \cap B) \subseteq Cl(A) \cap Cl(B)$.

(7) $Cl(A \cup B) = Cl(A) \cup Cl(B)$.

**Theorem 2.1.5** [18] Let $A$ and $B$ be subset of a topological space $(X, \tau)$. Then

(1) $Int(A) = A$.

(2) If $A \subseteq B$, then $Int(A) = A$.

(3) $A$ is open if and only if $Int(A) = A$.

(4) $Int(A)$ is the largest open in $X$ with $Int(A) \subseteq A$.

(5) $Int(Int(A)) = Int(A)$.

(6) $Int(A) \cap Int(B) = Int(A \cap B)$.

(7) $Int(A) \cup Int(B) \subseteq Int(A \cup B)$.

(8) If $A_\alpha \subseteq X$ for all $\alpha \in J$, then $\bigcup_{\alpha \in J} Int(A_\alpha) \subseteq Int\left(\bigcup_{\alpha \in J} A_\alpha\right)$.

**Definition 2.1.6** [10] A subset $A$ of a topological space $(X, \tau)$ is called:

(1) generalized closed (briefly, g-closed) if $Cl(A) \subseteq G$ whenever $A \subseteq G$ and $G$ is open in $X$;

(2) generalized open (briefly, g-open) if $X - A$ is g-closed.

**Definition 2.1.7** [14] A subset $A$ of a topological space $(X, \tau)$ is called mildly generalized closed (briefly, mildly g-closed) if $Cl(Int(A)) \subseteq G$ whenever $A \subseteq G$ and $G$ is a g-open set in $X$.

**Theorem 2.1.8** [14] For subsets $A, B$ of a topological space $(X, \tau)$, the following properties are hold:

(3) $A$ is mildly g-closed if and only if $Cl(Int(A)) - A$ contains no non-empty g-closed set.

(4) A mildly g-closed subset $A$ of $X$ is regular closed if and only if $Cl(Int(A)) - A$ is g-closed.

**Definition 2.1.9** [14] A subset $A$ of a topological space $(X, \tau)$ is called mildly generalized open (briefly, mildly g-open) if $X - A$ is a mildly g-closed.

**Theorem 2.1.11** [14] For subset $A$ of a topological space $(X,\tau)$, the following properties hold:

(1) $A$ is mildly g-open if and only if $F \subseteq Int(Cl(A))$, whenever $F \subseteq A$ and $F$ is a g-closed set.

(2) $A$ is mildly g-open if and only if $G = X$ , whenever $G$ is g-open and $Int(Cl(A)) \cup (X - A) \subseteq G$ .

(3) $A$ is mildly g-closed if and only if $Cl(Int(A)) - A$ is mildly g-open.

**Definition 2.1.10** [6] A triple $(X,\tau_1,\tau_2)$ where $X$ is a non-empty set and $\tau_1$ and $\tau_2$ are topologies on $X$ is called a bitopological space $(X,\tau_1,\tau_2)$ .

For a subset $A$ of a bitopological space $(X,\tau_1,\tau_2)$ , $\tau_i$-$Cl(A)$ (resp. $\tau_i$-$Int(A)$) denote the closure (resp. interior) of $A$ with respect to the topology $\tau_i$ for $i = 1,2$ .

**Definition 2.1.11** [1] A subset $A$ of a bitopological space $(X,\tau_1,\tau_2)$ is said to be $\tau_1\tau_2$-closed if $A = \tau_1$-$Cl(\tau_2$-$Cl(A))$. The complement of a $\tau_1\tau_2$-closed set is said to be $\tau_1\tau_2$-open .

**Definition 2.1.12** [1] Let $A$ be a subset of a bitopological space $(X,\tau_1,\tau_2)$ . Then

(1) The intersection of all $\tau_1\tau_2$ -closed sets containing $A$ is called $\tau_1\tau_2$-closure of $A$ and denoted by $\tau_1\tau_2$ - $Cl(A)$.

(2) The union of all $\tau_1\tau_2$ -open sets containing $A$ is called $\tau_1\tau_2$-interior of $A$ and denoted by $\tau_1\tau_2$ - $Int(A)$.

**Theorem 2.1.15** [1] Let $A$ and $B$ be subset of a bitopological space $(X,\tau_1,\tau_2)$ . For the $\tau_1\tau_2$-closure , the following properties hold;

(1) $A \subseteq \tau_1\tau_2$-$Cl(A)$ and $\tau_1\tau_2$-$Cl(\tau_1\tau_2$-$Cl(A)) = \tau_1\tau_2$-$Cl(A)$.

(2) If $A \subseteq B$, then $\tau_1\tau_2$-$Cl(A) \subseteq \tau_1\tau_2$-$Cl(B)$.

(3) $\tau_1\tau_2$-$Cl(A)$ is $\tau_1\tau_2$-closed .

(4) $A$ is $\tau_1\tau_2$-closed if and only if $A = \tau_1\tau_2$-$Cl(A)$.

(5) $\tau_1\tau_2$-$Cl(X - A) = X - \tau_1\tau_2$-$Int(X - A)$.

**Definition 2.1.16** [14] Let $(X,\tau)$ be a topological space. A subset $A$ of $X$ is called weakly generalized closed (briefly, weakly g-closed) if $Cl\left(Int(A)\right)\subseteq G$ whenever $A\subseteq G$ and $G$ is open in $X$.

## 2.2 Some properties of ideal topological spaces

In this section, we introduce some basic knowledge, definitions, notions and known propositions of ideal on topological and bitopological spaces that will be used in the next chapter.

**Definition 2.2.1** [8] A non-empty collection of subsets $I$ of a set $X$ is called an ideal on $X$ if satisfies the following properties:

(1) If $A\in I$ and $B\subseteq A$ implies $B\in I$.

(2) If $A\in I$ and $B\in I$ implies $A\cup B\in I$.

A topological space $(X,\tau)$ together with an ideal $I$ is called ideal topological space and is denoted by $(X,\tau,I)$.

**Definition 2.2.2** [5] A subset $A$ of an ideal topological space $(X,\tau,I)$ is said to be generalized closed with respect to an ideal (briefly, Ig-closed) if $Cl(A)-B\in I$ whenever $A\subseteq B$ and $B$ is an open set.

**Theorem 2.2.3** [5] For subsets $A$ and $B$ of an ideal topological space $(X,\tau,I)$, the following properties are hold:

(1) $A$ is Ig-closed if and only if $F\subseteq Cl(A)-A$ and $F$ is closed in $X$ implies $F\in I$.

(2) If $A$ and $B$ are Ig-closed, then their union $A\cup B$ is also Ig-closed.

(3) If $A$ is Ig-closed and $A\subseteq B\subseteq Cl(A)$, then $B$ is Ig-closed.

(4) If $A$ is Ig-closed and $F$ is closed, then $A\cap F$ is Ig-closed.

**Definition 2.2.4** [5] A subset $A$ of an ideal topological space $(X,\tau,I)$ is said to be generalized open with respect to an ideal (briefly, Ig-open) if $X-A$ is Ig-closed.

# CHAPTER 3

# WEAKLY GENERALIZED CLOSED SETS IN IDEAL TOPOLOGICAL SPACES

In this chapter, we introduce and study the notion of weakly generalized closed in topological spaces. Some properties of generalized closed sets with respect to an ideal obtained.

## 3.1 Weakly generalized closed sets in ideal topological spaces

We begin this section by introducing the concept of generalized closed sets with respect to an ideal.

**Definition 3.1.1** A subset $A$ of an ideal topological space $(X,\tau,I)$ is said to be *weakly generalized closed with respect to an ideal* (briefly, *wgI-closed*) if $Cl(Int(A))-U \in I$, whenever $A\subseteq U$ and $U$ is open in $X$.

**Remark 3.1.2** Every weakly g-closed set is wgI-closed, but the converse need not be true, as this may be seen from the following example.

**Example 3.1.3** Let $X=\{a,b,c\}$ with topology $\tau=\{\varnothing,\{b\},\{b,c\},X\}$ and ideal $I=\{\varnothing,\{a\},\{c\},\{a,c\}\}$. Then $A=\{b\}$ is wgI-closed,which is not weakly g-closed.

**Theorem 3.1.4** A subset $A$ of an ideal topological space $(X,\tau,I)$ is wgI-closed if and only if $F\subseteq Cl(Int(A))-A$ and $F$ is closed in $X$ implies $F\in I$.

**Proof.** Suppose that $A$ is wgI-closed set. Let $F$ be a closed set such that $F\subseteq Cl(Int(A))-A$. Then, we have $F\subseteq X-A$ and hence $A\subseteq X-F$. Since $A$ is wgI-closed and $X-F$ is open, $Cl(Int(A))-(X-F)\in I$. Since $Cl(Int(A))-(X-F)=Cl(Int(A))\cap F$ and $F\subseteq Cl(Int(A))$, we have $F\subseteq Cl(Int(A))\cap F$. Thus, $F\in I$.

Conversely, suppose that $F\subseteq Cl(Int(A))-A$ and $F$ is closed in $X$ implies $F\in I$. Let $U$ be an open set and $A\subseteq U$. Then, we have $X-U\subseteq X-A$, and hence

$$\begin{aligned} Cl(Int(A))-U &= Cl(Int(A))\cap(X-U) \\ &\subseteq Cl(Int(A))\cap(X-A) \\ &= Cl(Int(A))-A. \end{aligned}$$

Since $Cl(Int(A))\cap(X-U)$ is closed and by the hypothesis, $Cl(Int(A))-U\in I$. Consequently, we obtain $A$ is wgI-closed. ❑

**Remark 3.1.5** The union of two wgI-closed sets need not be a wgI-closed as shown by the following example.

**Example 3.1.6** Let $X=\{1,2,3\}$ with topology $\tau=\{\varnothing,\{2,3\},X\}$ and ideal $I=\{\varnothing,\{2\}\}$. Then $A=\{3\}$ and $B=\{2\}$ are wgI-closed sets, but $A\cup B=\{2,3\}$ is not wgI-closed.

**Proposition 3.1.7** Let $(X,\tau,I)$ be an ideal topological space and $A,B\subseteq X$. If $A$ and $B$ are both wgI-closed and open sets, then $A\cup B$ is also wgI-closed.

**Proof.** Suppose that $A$ and $B$ are both wgI-closed and open sets. Let $U$ be an open set such that $A\cup B\subseteq U$. Then, we have $A\subseteq U$ and $B\subseteq U$. Since $A$ and $B$ are wgI-closed, $Cl(Int(A))-U\in I$ and $Cl(Int(B))-U\in I$. Since $A$ and $B$ are open, we have $Cl(Int(A\cup B))-U=[Cl(Int(A))-U]\cup[Cl(Int(B))-U]\in I$ and hence $A\cup B$ is wgI-closed. ❑

**Proposition 3.1.8** Let $(X,\tau,I)$ be an ideal topological space and $A,B\subseteq X$. If $A$ is wgI-closed and $A\subseteq B\subseteq Cl(Int(A))$, then $B$ is wgI-closed.

**Proof.** Suppose that $A$ is wgI-closed and $A\subseteq B\subseteq Cl(Int(A))$. Let $U$ be an open set and $B\subseteq U$. Then, we have $A\subseteq U$. Since $A$ is wgI-closed, $Cl(Int(A))-U\in I$. Since $Cl(Int(A))=Cl(Int(B))$, we have $Cl(Int(B))-U=Cl(Int(A))-U\in I$ and hence $B$ is wgI-closed. ❑

**Remark 3.1.9** The intersection of two wgI-closed sets need not be a wgI-closed as shown by the following example.

**Example 3.1.10** Let $X=\{1,2,3\}$ with topology $\tau=\{\varnothing,\{1\},\{1,2\},X\}$ and ideal $I=\{\varnothing,\{3\}\}$. Then $A=\{1,2\}$ and $B=\{1,3\}$ are wgI-closed sets, but $A\cap B=\{1\}$ is not wgI-closed.

**Theorem 3.1.11** Let $(X,\tau,I)$ be an ideal topological space. If $A$ is a wgI-closed set and $F$ is a closed set, then $A\cap F$ is wgI-closed.

**Proof.** Suppose that $A$ is wgI-closed set and $F$ is closed set. Let $U$ be an open set and $A\cap F\subseteq U$. Then we have

$$X-U\subseteq X-(A\cap F)=(X-A)\cup(X-F)$$

and hence,

$$\begin{aligned} F\cap(X-U) &\subseteq F\cap[(X-A)\cup(X-F)] \\ &= F\cap(X-A) \\ &\subseteq X-A. \end{aligned}$$

Therefore, $A\subseteq X-[F\cap(X-U)]=U\cup(X-F)$. Since $A$ is wgI-closed and $U\cup(X-F)$ is open, $Cl(Int(A))-[U\cup(X-F)]\in I$. Since

$$\begin{aligned} Cl(Int(A\cap F)) &\subseteq Cl(Int(A)\cap Int(F)) \\ &\subseteq Cl(Int(A))\cap Cl(Int(F)) \\ &\subseteq Cl(Int(A))\cap Cl(F), \end{aligned}$$

we have

$$\begin{aligned} Cl(Int(A\cap F))-U &= Cl(Int(A\cap F))\cap(X-U) \\ &\subseteq [Cl(Int(A))\cap F]\cap(X-U) \\ &= Cl(Int(A))\cap[F\cap(X-U)] \\ &= Cl(Int(A))\cap[X-((X-F)\cup U)] \\ &= Cl(Int(A))-[(X-F)\cup U] \end{aligned}$$

and hence $Cl(Int(A\cap F))-U\in I$. Thus, $A\cap F$ is wgI-closed. ❑

**Definition 3.1.12** A subset $A$ of an ideal topological space $(X,\tau,I)$ is said to be *weakly generalized open with respect to an ideal* (briefly, *wgI-open*) if $X-A$ is wgI-closed.

**Example 3.1.13** Let $X=\{1,2,3\}$, $\tau=\{\varnothing,\{1\},\{1,2\},X\}$ and $I=\{\varnothing,\{2\}\}$. Then $\{1,3\}$ is wgI-open since $X-\{1,3\}=\{2\}$ is wgI-closed.

**Theorem 3.1.14** A subset $A$ of an ideal topological space $(X,\tau,I)$ is wgI-open if and only if $F-U\subseteq Int(Cl(A))$ for some $U\in I$, whenever $F\subseteq A$ and $F$ is closed.

**Proof.** Suppose that $A$ is wgI-open set. Let $F$ be a closed set and $F \subseteq A$. Then, we have $X - A \subseteq X - F$. Since $X - F$ is open and $X - A$ is wgI-closed, $Cl(Int(X - A)) - (X - F) \in I$. Thus, there exists $U \in I$ such that $U = Cl(Int(X - A)) - (X - F)$ and hence $Cl(Int(X - A)) \subseteq (X - F) \cup U$. Consequently, we obtain

$$F - U = X - [(X - F) \cup U] \subseteq X - Cl(Int(X - A)) = Int(Cl(A)).$$

Conversely, let $G$ be an open set and $X - A \subseteq G$. Then, we have $X - G \subseteq A$. By the hypothesis, $(X - G) - U \subseteq Int(Cl(A))$ for some $U \in I$. Therefore, $X - Int(Cl(A)) \subseteq X - [(X - G) - U]$ and hence $Cl(Int(X - A)) \subseteq G \cup U$ Since $Cl(Int(X - A)) - G = Cl(Int(X - A)) \cap (X - G) \subseteq (G \cup U) \cap (X - G)$ $= U \cap (X - G) \subseteq U$, we have $Cl(Int(X - A)) - G \in I$. Thus, $X - A$ is wgI-closed. This show that $A$ is wgI-open. ❑

Recall that the sets $A$ and $B$ are said to be separated if $Cl(A) \cap B = \varnothing$ and $Cl(B) \cap A = \varnothing$.

**Corollary 3.1.15** Let $(X, \tau, I)$ be an ideal topological space and $A, B \subseteq X$. If $A$ and $B$ are wgI-open and closed sets , then $A \cap B$ is wgI-open.

**Proof.** Suppose that $A$ and $B$ are wgI-open and closed. Then $X - A$ and $X - B$ are wgI-closed and open. By Proposition 3.1.8, we have $(X - A) \cup (X - B) = X - (A \cap B)$ is wgI-closed and so $A \cap B$ are wgI-open. ❑

**Example 3.1.16** Let $X = \{1,2,3\}$, $\tau = \{\varnothing, \{1\}, \{1,3\}, X\}$ and $I = \{\varnothing, \{2\}\}$. Let $A = \{2\}$ and $B = \{3\}$, then $A$ and $B$ are wgI-closed but $A \cup B = \{2,3\}$ is not wgI-open.

**Theorem 3.1.17** Let $(X, \tau, I)$ be an ideal topological space and $A, B \subseteq X$. If $A$ and $B$ are separated wgI-open sets, then $A \cup B$ is wgI-open.

**Proof.** Suppose that $A$ and $B$ are separated wgI-open sets. Let $F$ be a closed set and $F \subseteq A \cup B$. Then, we have $[F \cap Cl(A)] \subseteq A$ and $[F \cap Cl(B)] \subseteq B$. By the hypothesis, $[(F \cap Cl(A)) - U_1] \subseteq Int(Cl(A))$ and $[(F \cap Cl(B)) - U_2] \subseteq Int(Cl(B))$ for some $U_1, U_2 \in I$. Since

$$\left(F\cap Cl(A)\right)-Int\left(Cl(A)\right)\subseteq\left[\left(F\cap Cl(A)\right)\cup U_1\right]\cap\left[X-\left(Int\left(Cl(A)\right)\cup U_1\right)\right]\subseteq U_1$$

and

$$\left(F\cap Cl(B)\right)-Int\left(Cl(B)\right)\subseteq\left[\left(F\cap Cl(A)\right)\cup U_2\right]\cap\left[X-\left(Int\left(Cl(A)\right)\cup U_2\right)\right]\subseteq U_2,$$

$\left[\left(F\cap Cl(A)\right)-Int\left(Cl(A)\right)\right]\in I$ and $\left[\left(F\cap Cl(B)\right)-Int\left(Cl(B)\right)\right]\in I$ .

Therefore,

$$\left[\left(F\cap Cl(A)\right)-Int\left(Cl(A)\right)\right]\cup\left[\left(F\cap Cl(B)\right)-Int\left(Cl(B)\right)\right]\in I .$$

Since

$$\begin{aligned}&\left[F\cap\left(Cl(A)\cup Cl(B)\right)\right]-\left[Int\left(Cl(A)\right)\cup Int\left(Cl(B)\right)\right]\\ &\subseteq\left[\left(F\cap Cl(A)\right)-Int\left(Cl(A)\right)\right]\cup\left[\left(F\cap Cl(B)\right)-Int\left(Cl(B)\right)\right],\end{aligned}$$

$\left[F\cap\left(Cl(A)\cup Cl(B)\right)\right]-\left[Int\left(Cl(A)\right)\cup Int\left(Cl(B)\right)\right]\in I$ .

Since $F=F\cap(A\cup B)\subseteq F\cap Cl(A\cup B)$, we have

$$\begin{aligned}F-Int\left(Cl(A\cup B)\right)&\subseteq\left(F\cap Cl(A\cup B)\right)-Int\left(Cl(A\cup B)\right)\\ &\subseteq\left(F\cap Cl(A\cup B)\right)-\left[Int\left(Cl(A)\right)\cup Int\left(Cl(B)\right)\right]\end{aligned}$$

and hence $F-Int\left(Cl(A\cup B)\right)\in I$ . This implies that $F-U\subseteq Int\left(Cl(A\cup B)\right)$ for some $U\in I$ . Consequencely, we obtain $A\cup B$ is wgI-open. ❑

**Corollary 3.1.18** Let $(X,\tau,I)$ be an ideal topological space and $A,B\subseteq X$ . If $A$ and $B$ are wgI-closed sets such that $X-A$ and $X-B$ are separated, then $A\cap B$ is wgI-closed.

**Proof.** Suppose that $A$ and $B$ are wgI-closed sets. Then $X-A$ and $X-B$ are separated wgI-open. By Proposition 3.1.7, $(X-A)\cup(X-B)=X-(A\cap B)$ is wgI-open and so $A\cap B$ are wgI-closed. ❑

**Proposition 3.1.19** Let $(X,\tau,I)$ be an ideal topological space and $A,B\subseteq X$ . If $A$ is wgI-open set and $Int\left(Cl(A)\right)\subseteq B\subseteq A$, then $B$ is wgI-open.

**Proof.** Suppose that $A$ is wgI-open and $Int\left(Cl(A)\right)\subseteq B\subseteq A$. Then, we have $X-A\subseteq X-B\subseteq Cl\left(Int(X-A)\right)$ and by Proposition 3.1.10, $X-B$ is wgI-closed. Thus, $B$ is wgI-open in $X$ . ❑

**Theorem 3.1.20** Let $(X,\tau,I)$ be an ideal topological space and $A\subseteq X$. Then $A$ is wgI-closed if and only if $Cl(Int(A))-A$ is wgI-open.

**Proof.** Suppose that $A$ is a wgI-closed set. Let $F$ be a closed set and $F\subseteq Cl(Int(A))-A$. By Theorem 3.1.4, we have $F\in I$ and there exists $U\in I$ such that $U=F$. Thus, $F-U\subseteq Int\left[Cl\left[Cl(Int(A))-A\right]\right]$ and by Theorem 3.1.14, $Cl(Int(A))-A$ is wgI-open.

Conversely, suppose that $Cl(Int(A))-A$ is wgI-open. Let $G$ be an open set and $A\subseteq G$. Then, we have

$$\left[Cl(Int(A))\cap(X-G)\right]\subseteq Cl(Int(A))\cap(X-A)=Cl(Int(A))-A.$$

Since $Cl(Int(A))\cap(X-G)$ is closed and $Cl(Int(A))-A$ is wgI-open, by Theorem 3.1.11, $\left[Cl(Int(A))\cap(X-G)\right]-U\subseteq Cl(Int(A))-A$ for some $U\in I$ Since

$$\begin{aligned}
\left[Cl(Int(A))\cap(X-G)\right]-U &\subseteq Int\left[Cl\left[Cl(Int(A))-A\right]\right]\\
&= Int\left[Cl\left[Cl(Int(A))\cap(X-A)\right]\right]\\
&\subseteq Int\left[Cl(Cl(Int(A)))\cap Cl(X-A)\right]\\
&= Int(Cl(Int(A)))\cap Int(Cl(X-A))\\
&= Int(Cl(Int(A)))\cap\left[X-Cl(Int(A))\right]\\
&\subseteq Cl(Int(A))\cap\left[X-Cl(Int(A))\right]=\varnothing,
\end{aligned}$$

$Cl(Int(A))\cap(X-G)\subseteq U$ and hence $Cl(Int(A))-G\in I$. This shows that $A$ is wgI-closed. ❑

## 3.2 Some separation axioms

In this section, we introduce the notion of wgI-normal and wgI-regular spaces. Moreover, several interesting characterizations of these spaces are discussed.

**Definition 3.2.1** An ideal topological space $(X,\tau,I)$ is said to be *wgI-normal* if for every pair of disjoint wgI-closed sets $A$ and $B$ of $X$, there exist disjoint open sets $U$ and $V$ such that $A-U\in I$ and $B-V\in I$.

**Example 3.2.2** Let $X=\{a,b,c\}$ , $\tau=\{\varnothing,\{a\},\{b\},\{a,b\},\{a,c\},X\}$ and $I=\{\varnothing,\{b\},\{c\},\{b,c\}\}$. Then $(X,\tau,I)$ is wgI-normal.

The following theorem gives some characterizations of wgI-normal spaces.

**Theorem 3.2.3** For an ideal topological space $(X,\tau,I)$, the following are equivalent;

(1) $(X,\tau,I)$ is wgI-normal;

(2) for every wgI-closed set $F$ and wgI-open set $G$ containing $F$, there exists an open set $V$ such that $F-V\in I$ and $Cl(V)-G\in I$;

(3) for each pair of disjoint wgI-closed sets $A$ and $B$, there exists an open set $U$ such that $A-U\in I$ and $Cl(U)\cap B\in I$.

**Proof.** (1) $\Rightarrow$ (2) Suppose that $(X,\tau,I)$ is wgI-normal. Let $F$ be a wgI-closed set and $G$ be a wgI-open set such that $F\subseteq G$. Then, we have $X-G$ is a wgI-closed set and $(X-G)\cap F=\varnothing$. Since $(X,\tau,I)$ is wgI-normal, there exist disjoint open sets $U$ and $V$ such that $(X-G)-U\in I$ and $F-V\in I$. Since $U\cap V=\varnothing$, we have $V\subseteq X-U$ and hence $Cl(V)\subseteq Cl(X-U)=X-U$. Therefore,

$$(X-G)\cap Cl(V)\subseteq(X-G)\cap(X-U)$$

and hence

$$Cl(V)-G\subseteq(X-G)-U\in I.$$

This shows that, $Cl(V)-G\in I$.

(2) $\Rightarrow$ (3) Let $A$ and $B$ be disjoint wgI-closed sets of $X$. Then, we have $A\subseteq X-B$. By (2), there exists an open set $V$ such that $A-V\in I$ and $Cl(V)-(X-B)=Cl(V)\cap B\in I$.

(3) $\Rightarrow$ (1) Let $A$ and $B$ be disjoint wgI-closed sets in $X$. By (3), there exists an open set $U$ such that $A-U\in I$ and $Cl(U)\cap B\in I$. Now, $Cl(U)\cap B\in I$ implies that $B-[X-Cl(U)]\in I$. Put $V=X-Cl(U)$, then $V$ is an open set such that $B-V\in I$ and $U\cap V=U\cap[X-Cl(U)]=\varnothing$. Hence, $(X,\tau,I)$ is wgI-normal. ❑

**Definition 3.2.4** An ideal topological space $(X,\tau,I)$ is said to be *wgI-regular*, if for each wgI-closed set $F$ such that $x\notin F$, there exist disjoint open sets $U$ and $V$ such that $x\in U$ and $F-V\in I$.

**Example 3.2.5** Let $X=\{a,b\}$ , $\tau=\{\varnothing,\{a\},\{b\},X\}$ and $I=\{\varnothing,\{b\}\}$. Then $(X,\tau,I)$ is wgI-regular.

The following theorem gives some characterizations of wgI-regular spaces.

**Theorem 3.2.6** Let $(X,\tau,I)$ be an ideal topological space. Then the following are equivalent;

(1) $(X,\tau,I)$ is wgI-regular;

(2) for each $x\in X$ and wgI-open set $U$ containing $x$, there exists an open set $V$ containing $x$ such that $Cl(Int(V))-U\in I$;

(3) for each $x\in X$ and wgI-closed sets $A$ not containing $x$, there exists an open set $V$ containing $x$ such that $Cl(Int(V))\cap A\in I$.

**Proof.** (1) $\Rightarrow$ (2) Suppose that $(X,\tau,I)$ is wgI-regular. Let $x\in X$ and $U$ be a wgI-open set containing $x$. Then, we have $x\notin X-U$ and by (1), there exist disjoint open sets $V$ and $W$ such that $x\in V$ and $(X-U)-W\in I$. Since $(X-U)-W=I\in I$, there exists $I_0\in I$ such that $(X-U)-W=I_0\in I$. Then, we have $X-U=W\cup I_0$. Now, $V\cap W=\varnothing$ implies that $V\subseteq X-W$ and $Cl(Int(V))\subseteq X-W$. Therefore,

$$\begin{aligned} Cl(Int(V))-U &= Cl(Int(V))\cap(X-U) \\ &\subseteq (X-W)\cap(W\cup I_0) \\ &= (X-W)\cap I_0 \\ &\subseteq I_0\in I. \end{aligned}$$

(2) $\Rightarrow$ (3) Let $x\in X$ and $F$ be a wgI-closed set not containing $x$. Then, we have $x\in X-F$ and by (2), there exists an open set $V$ containing $x$ such that $Cl(Int(V))-(X-F)\in I$ and so $Cl(Int(V))\cap F\in I$.

(3) $\Rightarrow$ (1) Let $F$ be a wgI-closed set such that $x\in F$. By (3), there exists anopen set $V$ containing $x$ such that $Cl(Int(V))\cap F\in I$. If $Cl(Int(V))\cap F=I_0\in I$, then $F-[X-Cl(Int(V))]=I_0\in I$. $V$ and $X-Cl(Int(V))$ are the required disjoint open sets such that $x\in V$ and $F-[X-Cl(Int(V))]\in I$. Consequently, we obtain $(X,\tau,I)$ is wgI-regular. ❑

# CHAPTER 4

## WEAKLY GENERALIZED $\tau_1\tau_2$-CLOSED SETS WITH RESPECT TO AN IDEAL IN BITOPOLOGICAL SPACES

In this chapter, we introduce and study the notion of weakly generalized $\tau_1\tau_2$-closed sets in ideal bitopological spaces. Some properties of generalized $\tau_1\tau_2$-closed sets and discussed.

### 4.1 Weakly generalized $\tau_1\tau_2$-closed sets in bitopological spaces

In this section, we introduce the notion of weakly generalized $\tau_1\tau_2$-closed sets in ideal bitopological spaces and investigate some of their properties.

**Definition 4.1.1** A subset $A$ of a bitopological space $(X,\tau_1,\tau_2)$ is said to be *weakly generalized* $\tau_1\tau_2$*-closed* (briefly, $wg\text{-}\tau_1\tau_2\text{-}closed$) if $\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Int(A)) \subseteq U$, whenever $A \subseteq U$ and $U$ is $\tau_1\tau_2$-open.

**Example 4.1.2** Let $X=\{1,2,3\}$, $\tau_1=\{\varnothing,\{1\},\{2\},\{1,2\},X\}$ and $\tau_2=\{\varnothing,\{2\},\{3\},\{2,3\},X\}$. Then $A=\{1,3\}$ is wg-$\tau_1\tau_2$-closed.

**Proposition 4.1.3** Every $\tau_1\tau_2$-closed sets is wg-$\tau_1\tau_2$-closed.

**Proof** Suppose that $A$ is a $\tau_1\tau_2$-closed set. Let $U$ be a $\tau_1\tau_2$-open set and $A \subseteq U$. Then $\tau_1\tau_2\text{-}Cl(A) \subseteq U$ and hence $\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Int(A)) \subseteq \tau_1\tau_2\text{-}Cl(Int(A)) \subseteq U$. Thus, $\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Int(A)) \subseteq U$ Therefore, $A$ is wg-$\tau_1\tau_2$-closed. ❑

**Remark 4.1.4** The converse of the above proposition need not be true, as this may be seen from the following example.

**Example 4.1.5** In the example 4.1.2, Let $A=\{1,2\}$ is wg-$\tau_1\tau_2$-closed, but $A$ is not $\tau_1\tau_2$-closed in $(X,\tau_1,\tau_2)$.

**Proposition 4.1.6** Let $(X,\tau_1,\tau_2)$ be a bitopological space and $A,B\subseteq X$. If $A$ and $B$ are both $\tau_1\tau_2$-open and wg-$\tau_1\tau_2$-closed sets, then $A\cup B$ is also wg-$\tau_1\tau_2$-closed.

**Proof.** Suppose that $A$ and $B$ are both $\tau_1\tau_2$-open and wg-$\tau_1\tau_2$-closed sets. Let $U$ be a $\tau_1\tau_2$-open set such that $A\cup B\subseteq U$. Then, we have $A\subseteq U$ and $B\subseteq U$. Since $A$ and $B$ are wg-$\tau_1\tau_2$-closed, $\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Int(A))\subseteq U$ and $\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Int(B))\subseteq U$. Since $A$ and $B$ are $\tau_1\tau_2$-open, we have

$$\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Int(A\cup B))=[\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Int(A))]\cup[\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Int(B))]\in I$$

and hence $A\cup B$ is wg-$\tau_1\tau_2$-closed. ❑

**Remark 4.1.7** The union of two wg-$\tau_1\tau_2$-closed sets need not be a wg-$\tau_1\tau_2$-closed set as shown by the following example.

**Example 4.1.8** Let $X=\{1,2,3,4\}$, $\tau_1=\{\varnothing,\{1\},\{1,2,3\},X\}$ and $\tau_2=\{\varnothing,\{2\},\{1,2,3\},X\}$. Then $A=\{1,2\}$ and $B=\{3\}$ are wg-$\tau_1\tau_2$-closed but $A\cup B=\{1,2,3\}$ is not wg-$\tau_1\tau_2$-closed.

**Theorem 4.1.9** Let $(X,\tau_1,\tau_2)$ be a bitopological space. If $A$ is a wg-$\tau_1\tau_2$-closed set and $F$ is a $\tau_1\tau_2$-closed set, then $A\cap F$ is wg-$\tau_1\tau_2$-closed.

**Proof.** Suppose that $A$ is a wg-$\tau_1\tau_2$-closed set and $F$ is a $\tau_1\tau_2$-closed set. Let $U$ be a $\tau_1\tau_2$-open set such that $A\cap F\subseteq U$. Since $A\cap F\subseteq U$, we have $A\subseteq U\cup(X-F)$. Since $A$ is wg-$\tau_1\tau_2$-closed and $U\cup(X-F)$ is $\tau_1\tau_2$-open, $\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Int(A))\subseteq U\cup(X-F)$ and hence,

$$\begin{aligned}[\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Int(A))\cap F] &\subseteq [U\cup(X-F)]\cap F\\ &= (U\cap F)\cup[(X-F)\cap F]\\ &= U.\end{aligned}$$

Since $\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Int(A\cap F))\subseteq\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Int(A))\cap F$, $\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Int(A\cap F))\subseteq U$. Hence, $A\cap F$ is wg-$\tau_1\tau_2$-closed. ❑

**Remark 4.1.10** The intersection of a $\tau_1\tau_2$-closed set and a wg-$\tau_1\tau_2$-closed set need not be a wg-$\tau_1\tau_2$-closed set as shown by the following example.

**Example 4.1.11** Let $X=\{1,2,3\}$, $\tau_1=\{\varnothing,\{1\},X\}$ and $\tau_2=\{\varnothing,\{1\},X\}$. Then $A=\{1,3\}$ and $B=\{1,2\}$ are wg-$\tau_1\tau_2$-closed but $A\cap B=\{1\}$ is not wg-$\tau_1\tau_2$-closed.

**Theorem 4.1.12** Let $(X,\tau_1,\tau_2)$ be a bitopological space and $A\subseteq X$. Then $A$ is wg-$\tau_1\tau_2$-closed if and only if $\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Int(A))-A$ contains no non-empty $\tau_1\tau_2$-closed set.

**Proof.** $(\Rightarrow)$ Let $F$ be a $\tau_1\tau_2$-closed set such that $F\subseteq\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Int(A))-A$, $F\subseteq\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Int(A))\cap(X-A)$. Since $X-F$ is $\tau_1\tau_2$-open and $A\subseteq(X-F)$, $\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Int(A))\subseteq(X-F)$ and so $F\subseteq X-(\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Int(A)))$. This implies that $F\subseteq[\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Int(A))]\cap(X-[\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Int(A))])=\varnothing$. Therefore, $F=\varnothing$.

$(\Leftarrow)$ Let $G$ be a $\tau_1\tau_2$-open set such that $A\subseteq G$. Suppose that $\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Int(A))\not\subseteq G$. Then $\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Int(A))\cap(X-G)\neq\varnothing$. Therefore, $\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Int(A))\cap(X-G)$ is $\tau_1\tau_2$-closed and

$$\begin{aligned}\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Int(A))\cap(X-G) &\subseteq \tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Int(A))\cap(X-A)\\ &= \tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Int(A))-A\end{aligned}$$

This is a contradiction. Hence, $\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Int(A))\subseteq G$. This shows that, $A$ is wg-$\tau_1\tau_2$-closed. ❑

**Proposition 4.1.13** Let $(X,\tau_1,\tau_2)$ be a bitopological space and $A,B\subseteq X$. If $A$ is a wg-$\tau_1\tau_2$-closed set and $A\subseteq B\subseteq\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Int(A))$, then $B$ is a wg-$\tau_1\tau_2$-closed set.

**Proof** Suppose that $A$ is a wg-$\tau_1\tau_2$-closed set and $A\subseteq B\subseteq\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Int(A))$. Then, we have $[\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Int(B))]-B\subseteq[\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Int(A))]-A$. By Theorem 4.1.12, $[\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Int(A))]-A$ contains no non-empty $\tau_1\tau_2$-closed set and so $[\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Int(B))]-B$. Again, by Theorem 4.1.12, $B$ is wg-$\tau_1\tau_2$-closed. ❑

**Definition 4.1.14** A subset $A$ of a bitopological space $(X,\tau_1,\tau_2)$ is said to be *weakly generalized* $\tau_1\tau_2$*-open* (briefly, *wg*-$\tau_1\tau_2$-*open*) if $X-A$ is wg-$\tau_1\tau_2$-closed.

**Example 4.1.15** In the example 4.1.2, $\{2\}$ is wg-$\tau_1\tau_2$-open.

**Theorem 4.1.16** A subset $A$ of a bitopological space $(X,\tau_1,\tau_2)$ is wg-$\tau_1\tau_2$-open if and only if $F \subseteq \tau_1\tau_2\text{-}Int(\tau_1\tau_2\text{-}Cl(A))$ whenever $F \subseteq A$ and $F$ is $\tau_1\tau_2$-closed.

**Proof** $(\Rightarrow)$ Let $A$ be a wg-$\tau_1\tau_2$-open set and $F$ be a $\tau_1\tau_2$-closed set such that $F \subseteq A$. Since $X-A$ is wg-$\tau_1\tau_2$-closed and $X-F$ is $\tau_1\tau_2$-open, $\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Int(A)) \subseteq X-F$, $F \subseteq \tau_1\tau_2\text{-}Int(\tau_1\tau_2\text{-}Cl(A))$.

$(\Leftarrow)$ Let $G$ be a $\tau_1\tau_2$-open set such that $X-A \subseteq G$. Then $X-G \subseteq A$ and hence $X-G \subseteq \tau_1\tau_2\text{-}Int(\tau_1\tau_2\text{-}Cl(A))$. It follows that $\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Int(X-A)) \subseteq G$. Therefore, $X-A$ is wg-$\tau_1\tau_2$-closed and so $A$ is wg-$\tau_1\tau_2$-open. ❑

## 4.2 Weakly generalized $\tau_1\tau_2$ -closed sets with respect to an ideal

In this section, we introduce the notion of weakly generalized $\tau_1\tau_2$ -closed sets with respect to an ideal in bitopological spaces and investigate some properties of these sets.

**Definition 4.2.1** A subset $A$ of an ideal bitopological space $(X,\tau_1,\tau_2,I)$ is said to be *weakly generalized* $\tau_1\tau_2$ *-closed set with respect to an ideal* (briefly, *wgI* - $\tau_1\tau_2$ - *closed*) if $\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Int(A))-U \in I$, whenever $A \subseteq U$ and $U$ is $\tau_1\tau_2$-open.

**Example 4.2.2** Let $X=\{a,b,c\}$ with topologies $\tau_1=\{\varnothing,\{b\},\{b,c\},X\}$, $\tau_2=\{\varnothing,\{b\},\{a,b\},X\}$ and ideal $I=\{\varnothing,\{a,c\}\}$. Then $A=\{b\}$ is wgI-$\tau_1\tau_2$-closed but not wg-$\tau_1\tau_2$-closed.

**Remark 4.2.3** The union of two wgI-$\tau_1\tau_2$-closed sets need not be a wgI-$\tau_1\tau_2$-closed set as shown by the following example.

**Example 4.2.4** Let $X=\{1,2,3\}$, $\tau_1=\{\varnothing,\{2,3\},X\}$, $\tau_2=\{\varnothing,\{2\},\{2,3\},X\}$ and $I=\{\varnothing,\{2\}\}$. If $A=\{3\}$ and $B=\{2\}$, then $A$ and $B$ are wgI-$\tau_1\tau_2$-closed but $A\cup B=\{2,3\}$ is not wgI-$\tau_1\tau_2$-closed.

**Proposition 4.2.5** Let $(X,\tau_1,\tau_2,I)$ be an ideal bitopological space and $A,B\subseteq X$. If $A$ and $B$ are both wgI-$\tau_1\tau_2$-closed and $\tau_1\tau_2$-open sets, then $A\cup B$ is also wgI-$\tau_1\tau_2$-closed.

**Proof.** Suppose that $A$ and $B$ are both wgI-$\tau_1\tau_2$-closed and $\tau_1\tau_2$-open sets. Let $U$ be a $\tau_1\tau_2$-open set such that $A\cup B\subseteq U$. Then, we have $A\subseteq U$ and $B\subseteq U$. Since $A$ and $B$ are wgI-$\tau_1\tau_2$-closed, $\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Int(A))-U\in I$ and $\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Int(B))-U\in I$. Since $A$ and $B$ are $\tau_1\tau_2$-open, we have

$$\begin{aligned}&\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Int(A\cup B))-U\\ &=\left[\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Int(A))-U\right]\cup\left[\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Int(B))-U\right]\end{aligned}$$

Thus, $\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Int(A\cup B))-U\in I$ and hence $A\cup B$ is wgI-$\tau_1\tau_2$-closed. ❑

The following theorem gives some properties of wgI-$\tau_1\tau_2$-closed sets.

**Theorem 4.2.6** A subset $A$ of an ideal bitopological space $(X,\tau_1,\tau_2,I)$ is wgI-$\tau_1\tau_2$-closed if and only if $F\subseteq\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-Int}(A))-A$ and $F$ is $\tau_1\tau_2$-closed in $X$ implies $F\in I$.

**Proof.** Suppose that $A$ is wgI-$\tau_1\tau_2$-closed set. Let $F$ be a $\tau_1\tau_2$-closed set such that $F\subseteq\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-Int}(A))-A$. Then, we have $F\subseteq X-A$ and hence $A\subseteq X-F$. Since $A$ is wgI-$\tau_1\tau_2$-closed and is $X-F$ is $\tau_1\tau_2$-open, $\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-Int}(A))-(X-F)\in I$. Since

$$\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Int(A))-(X-F)=\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Int(A))\cap F$$

and $F\subseteq\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Int(A))$, we have $F\subseteq\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Int(A))\cap F$. Thus, $F\in I$.

Conversely, suppose that $F\subseteq\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Int(A))-A$ and $F$ is $\tau_1\tau_2$-closed in $X$ implies $F\in I$. Let $U$ be a $\tau_1\tau_2$-open set such that $A\subseteq U$. Then, we have $X-U\subseteq X-A$ and hence

$$\begin{aligned}\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Int(A))-U &= \tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Int(A))\cap(X-U)\\ &\subseteq \tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Int(A))\cap(X-A)\\ &= \tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Int(A))-A.\end{aligned}$$

Since $\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Int(A))\cap(X-U)$ is $\tau_1\tau_2$-closed and by the hypothesis, $\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Int(A))-U\in I$. Consequently, we obtain $A$ is wgI-$\tau_1\tau_2$-closed. ❑

**Proposition 4.2.7** Let $(X,\tau_1,\tau_2,I)$ be an ideal bitopological space and $A,B\subseteq X$. If $A$ is wgI-$\tau_1\tau_2$-closed and $A\subseteq B\subseteq \tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Int(A))$, then $B$ is wgI-$\tau_1\tau_2$-closed.

**Proof.** Suppose that $A$ is wgI-$\tau_1\tau_2$-closed and $A\subseteq B\subseteq \tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Int(A))$. Let $U$ be a $\tau_1\tau_2$-open set and $B\subseteq U$. Then, we have $A\subseteq U$. Since $A$ is wgI-$\tau_1\tau_2$-closed, $\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Int(A))-U\in I$. Since

$$\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Int(A))=\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Int(B)),$$

we have

$$\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Int(B))-U=\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Int(A))-U\in I$$

and hence $B$ is wgI-$\tau_1\tau_2$-closed. ❑

**Remark 4.2.8** The intersection of two wgI-$\tau_1\tau_2$-closed sets need not be a wgI-$\tau_1\tau_2$-closed set as shown by the following example.

**Example 4.2.9** Let $X=\{1,2,3\}$ with topologies $\tau_1=\{\varnothing,\{1\},\{2\},\{1,2\},X\}$, $\tau_2=\{\varnothing,\{1\},\{2\},\{1,2\},\{1,3\},X\}$ and ideal $I=\{\varnothing,\{2\}\}$. Then $A=\{1,2\}$ and $B=\{2,3\}$ are wgI-$\tau_1\tau_2$-closed, but $A\cap B=\{2\}$ is not wgI-$\tau_1\tau_2$-closed.

**Corollary 4.2.10** If $A$ and $B$ are wgI-$\tau_1\tau_2$-open and $\tau_1\tau_2$-closed sets, then $A\cap B$ is wgI-$\tau_1\tau_2$-open.

**Proof.** Suppose that $A$ and $B$ are wgI-$\tau_1\tau_2$-open and $\tau_1\tau_2$-closed set. Then $X-A$ and $X-B$ are wgI-$\tau_1\tau_2$-open and $\tau_1\tau_2$-open. By Theorem 4.2.5, we have $(X-A)\cup(X-B)=X-(A\cap B)$ is wgI-$\tau_1\tau_2$-closed and so $A\cap B$ is wgI-$\tau_1\tau_2$-open. ❑

**Theorem 4.2.11** Let $(X,\tau_1,\tau_2,I)$ be an ideal bitopological space. If $A$ be a wgI-$\tau_1\tau_2$-closed set and $F$ be a $\tau_1\tau_2$-closed set, then $A\cap F$ is wgI-$\tau_1\tau_2$-closed.

**Proof.** Suppose that $A$ is a wgI-$\tau_1\tau_2$-closed set and $F$ be a $\tau_1\tau_2$-closed set. Let $U$ be a $\tau_1\tau_2$-open set and $A\cap F\subseteq U$. Then, we have

$$X-U\subseteq X-(A\cap F)=(X-A)\cup(X-F)$$

and hence

$$\begin{aligned} F\cap(X-U) &\subseteq F\cap\left[(X-A)\cup(X-F)\right] \\ &= F\cap(X-A) \\ &\subseteq X-A. \end{aligned}$$

Therefore, $A\subseteq X-\left[F\cap(X-U)\right]=U\cup(X-F)$. Since $A$ is wgI-$\tau_1\tau_2$-closed and $U\cup(X-F)$ is $\tau_1\tau_2$-open, $\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Int(A))-\left[U\cup(X-F)\right]\in I$. Since

$$\begin{aligned} \tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Int(A\cap F)) &\subseteq \tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Int(A)\cap\tau_1\tau_2\text{-}Int(F)) \\ &\subseteq \tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Int(A))\cap\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Int(F)) \\ &\subseteq \tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Int(A))\cap F, \\ \tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Int(A\cap F))-U &= \tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Int(A\cap F))\cap(X-U) \\ &\subseteq \left[\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Int(A))\cap F\right]\cap(X-U) \\ &= \tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Int(A))\cap\left[F\cap(X-U)\right] \\ &= \tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Int(A))\cap\left[X-((X-F)\cup U)\right] \\ &= \tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Int(A))-\left[(X-F)\cup U\right] \end{aligned}$$

and hence $\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Int(A\cap F))-U\in I$. Thus, $A\cap F$ is wgI-$\tau_1\tau_2$-closed. ❑

**Definition 4.2.12** A subset $A$ of an ideal bitopological space $(X,\tau_1,\tau_2,I)$ is said to be *weakly generalized open with respect to an ideal* (briefly, *wgI* - $\tau_1\tau_2$ - *open*) if $X-A$ is wgI-$\tau_1\tau_2$-closed.

**Example 4.2.13** In Example 4.2.2, $\{a,c\}$ is wgI-$\tau_1\tau_2$-open.

**Theorem 4.2.14** A subset $A$ of an ideal bitopological space $(X,\tau_1,\tau_2,I)$ is wgI-$\tau_1\tau_2$-open if and only if $F-U\subseteq\tau_1\tau_2\text{-}Int(\tau_1\tau_2\text{-}Cl(A))$ for some $U\in I$ whenever $F\subseteq A$ and $F$ is $\tau_1\tau_2$-closed.

**Proof.** Suppose that $A$ is wgI-$\tau_1\tau_2$-open. Let $F$ be a $\tau_1\tau_2$-closed set and $F\subseteq A$. Then, we have $X-A\subseteq X-F$. Since $X-F$ is $\tau_1\tau_2$-open and $X-A$ is wgI-$\tau_1\tau_2$-closed, $\left[\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Int(X-A))\right]-(X-F)\in I$. Thus, there exists $U\in I$ such that $U=\left[\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Int(X-A))\right]-(X-F)$ and hence

$$\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Int(X-A))\subseteq(X-F)\cup U.$$

Consequently, we obtain

$$F-U=X-\left[(X-F)\cup U\right]\subseteq X-\left[\tau_1\tau_2\text{-}Cl\left(\tau_1\tau_2\text{-}Int(X-A)\right)\right]=\tau_1\tau_2\text{-}Int\left(\tau_1\tau_2\text{-}Cl(A)\right).$$

Conversely, let $G$ be a $\tau_1\tau_2$-open set and $X-A\subseteq G$. Then, we have $X-G\subseteq A$. By the hypothesis, $(X-G)-U\subseteq\tau_1\tau_2\text{-}Int\left(\tau_1\tau_2\text{-}Cl(A)\right)$ for some $U\in I$. Therefore, $X-\left[\tau_1\tau_2\text{-}Int\left(\tau_1\tau_2\text{-}Cl(A)\right)\right]\subseteq X-\left[(X-G)-U\right]$ and hence

$$\tau_1\tau_2\text{-}Cl\left(\tau_1\tau_2\text{-}Int(X-A)\right)\subseteq G\cup U.$$

Since,

$$\begin{aligned}\tau_1\tau_2\text{-}Cl\left(\tau_1\tau_2\text{-}Int(X-A)\right)-G &= \tau_1\tau_2\text{-}Cl\left(\tau_1\tau_2\text{-}Int(X-A)\right)\cap(X-G)\\ &\subseteq (G\cup U)\cap(X-G)\\ &= U\cap(X-G)\\ &\subseteq U,\end{aligned}$$

$\left[\tau_1\tau_2\text{-}Cl\left(\tau_1\tau_2\text{-}Int(X-A)\right)\right]-G\in I$. Thus, $X-A$ is wgI-$\tau_1\tau_2$-closed. This show that $A$ is wgI-$\tau_1\tau_2$-open. ❑

**Definition 4.2.15** Let $(X,\tau_1,\tau_2,I)$ be an ideal bitopological space and $A,B\subseteq X$. Then $A$ and $B$ are said to be separated if $\tau_1\tau_2\text{-}Cl(A)\cap B=\varnothing$ and $\tau_1\tau_2\text{-}Cl(B)\cap A=\varnothing$.

**Example 4.2.16** Let $X=\{1,2,3\}$, $\tau_1=\{\varnothing,\{1\},\{2\},\{1,2\},X\}$, $\tau_2=\{\varnothing,\{2\},\{1,2\},X\}$ and $I=\{\varnothing,\{3\}\}$. If $A=\{1\}$ and $B=\{2\}$, then $A$ and $B$ are wgI-$\tau_1\tau_2$-closed sets, but $A\cup B=\{1,3\}$ is not wgI-$\tau_1\tau_2$-open.

**Theorem 4.2.17** Let $(X,\tau_1,\tau_2,I)$ be an ideal bitopological space and $A,B\subseteq X$. If $A$ and $B$ are separated wgI-$\tau_1\tau_2$-open sets, then $A\cup B$ is wgI-$\tau_1\tau_2$-open.

**Proof.** Suppose that $A$ and $B$ are separated wgI-$\tau_1\tau_2$-open sets. Let $F$ be a $\tau_1\tau_2$-closed set and $F\subseteq A\cup B$. Then, we have $\left[F\cap\tau_1\tau_2\text{-}Cl(A)\right]\subseteq A$ and $\left[F\cap\tau_1\tau_2\text{-}Cl(B)\right]\subseteq B$. By the hypothesis,

$$\left[\left(F\cap\tau_1\tau_2\text{-}Cl(A)\right)-U_1\right]\subseteq\tau_1\tau_2\text{-}Int\left(\tau_1\tau_2\text{-}Cl(A)\right)$$

and

$$\left[\left(F\cap\tau_1\tau_2\text{-}Cl(B)\right)-U_2\right]\subseteq\tau_1\tau_2\text{-}Int\left(\tau_1\tau_2\text{-}Cl(B)\right)\text{ for some }U_1,U_2\in I.$$

Since

$$\left[\left(F\cap\tau_1\tau_2\text{-}Cl(A)\right)\right]-\left[\tau_1\tau_2\text{-}Int\left(\tau_1\tau_2\text{-}Cl(A)\right)\right]$$
$$\subseteq\left[\left(F\cap\tau_1\tau_2\text{-}Cl(A)\right)\cup U_1\right]\cap\left[\left(F\cap\tau_1\tau_2\text{-}Cl(A)\right)\cup U_1\right]\subseteq U_1$$

and

$$\left[\left(F\cap\tau_1\tau_2\text{-}Cl(B)\right)\right]-\left[\tau_1\tau_2\text{-}Int\left(\tau_1\tau_2\text{-}Cl(B)\right)\right]$$
$$\subseteq\left[\left(F\cap\tau_1\tau_2\text{-}Cl(B)\right)\cup U_2\right]\cap\left[\left(F\cap\tau_1\tau_2\text{-}Cl(B)\right)\cup U_2\right]\subseteq U_2,$$
$$\left[\left(F\cap\tau_1\tau_2\text{-}Cl(A)\right)\right]-\left[\tau_1\tau_2\text{-}Int\left(\tau_1\tau_2\text{-}Cl(A)\right)\right]\in I$$

and

$$\left[\left(F\cap\tau_1\tau_2\text{-}Cl(B)\right)\right]-\left[\tau_1\tau_2\text{-}Int\left(\tau_1\tau_2\text{-}Cl(B)\right)\right]\in I\ .$$

Therefore,

$$\left[\left(F\cap\tau_1\tau_2\text{-}Cl(A)\right)-\tau_1\tau_2\text{-}Int\left(\tau_1\tau_2\text{-}Cl(A)\right)\right]\cup\left[\left(F\cap\tau_1\tau_2\text{-}Cl(B)\right)-\tau_1\tau_2\text{-}Int\left(\tau_1\tau_2\text{-}Cl(B)\right)\right]\in I\ .$$

Since

$$\left[F\cap\left(\tau_1\tau_2\text{-}Cl(A)\cup\tau_1\tau_2\text{-}Cl(B)\right)\right]-\left[\tau_1\tau_2\text{-}Int\left(\tau_1\tau_2\text{-}Cl(A)\right)\cup\tau_1\tau_2\text{-}Int\left(\tau_1\tau_2\text{-}Cl(B)\right)\right]$$
$$\subseteq\left[\left(F\cap\tau_1\tau_2\text{-}Cl(A)\right)-\tau_1\tau_2\text{-}Int\left(\tau_1\tau_2\text{-}Cl(A)\right)\right]\cup\left[\left(F\cap\tau_1\tau_2\text{-}Cl(B)\right)-\tau_1\tau_2\text{-}Int\left(\tau_1\tau_2\text{-}Cl(B)\right)\right],$$
$$\left[F\cap\left(\tau_1\tau_2\text{-}Cl(A)\cap\tau_1\tau_2\text{-}Cl(B)\right)\right]-\left[\tau_1\tau_2\text{-}Int\left(\tau_1\tau_2\text{-}Cl(A)\right)\cup\tau_1\tau_2\text{-}Int\left(\tau_1\tau_2\text{-}Cl(B)\right)\right]\in I.$$

Since $F=F\cap(A\cup B)\subseteq F\cup\tau_1\tau_2\text{-}Cl(A\cup B)$, we have

$$F-\tau_1\tau_2\text{-}Int\left(\tau_1\tau_2\text{-}Cl(A\cup B)\right)$$
$$\subseteq\left(F\cap\tau_1\tau_2\text{-}Cl(A\cup B)\right)-\tau_1\tau_2\text{-}Int\left(\tau_1\tau_2\text{-}Cl(A\cup B)\right)$$
$$\subseteq\left(F\cap\tau_1\tau_2\text{-}Cl(A\cup B)\right)-\left[\tau_1\tau_2\text{-}Int\left(\tau_1\tau_2\text{-}Cl(A)\right)\cup\tau_1\tau_2\text{-}Int\left(\tau_1\tau_2\text{-}Cl(B)\right)\right]$$

and hence $F-\tau_1\tau_2\text{-}Int\left(\tau_1\tau_2\text{-}Cl(A\cup B)\right)\in I$.

This implies that $F-U\subseteq\tau_1\tau_2\text{-}Int\left(\tau_1\tau_2\text{-}Cl(A\cup B)\right)$ for some $U\in I$. Consequently, we obtain $A\cup B$ is wgI-$\tau_1\tau_2$-open. ❑

**Proposition 4.2.18** Let $(X,\tau_1,\tau_2,I)$ be an ideal bitopological space and $A,B\subseteq X$. If $A$ is a wgI-$\tau_1\tau_2$-open set and $\tau_1\tau_2\text{-}Int\left(\tau_1\tau_2\text{-}Cl(A)\right)\subseteq B\subseteq A$, then $B$ is wgI-$\tau_1\tau_2$-open.

**Proof.** Suppose that $A$ is wgI-$\tau_1\tau_2$-open and $\tau_1\tau_2$-$Int(\tau_1\tau_2$-$Cl(A)) \subseteq B \subseteq A$. Then, we have $X - A \subseteq X - B \subseteq \tau_1\tau_2$-$Cl(\tau_1\tau_2$-$Int(X - A))$ and by Proposition 4.2.7, $X - B$ is wgI-$\tau_1\tau_2$-closed. Thus, $B$ is wgI-$\tau_1\tau_2$-open. ❑

**Corollary 4.2.19** Let $(X, \tau_1, \tau_2, I)$ be an ideal bitopological space and $A, B \subseteq X$. If $A$ and $B$ are wgI-$\tau_1\tau_2$-closed sets such that $X - A$ and $X - B$ are separated, then $A \cap B$ is wgI-$\tau_1\tau_2$-closed.

**Proof.** Suppose that $A$ and $B$ are wgI-$\tau_1\tau_2$-closed sets. Then $X - A$ and $X - B$ are separated wgI-$\tau_1\tau_2$-open. By Proposition 4.2.18, $(X - A) \cup (X - B) = X - (A \cap B)$ is wgI-$\tau_1\tau_2$-open and so $A \cap B$ is wgI-$\tau_1\tau_2$-closed. ❑

**Theorem 4.2.20** Let $(X, \tau_1, \tau_2, I)$ be an ideal bitopological space and $A \subseteq X$. Then $A$ is wgI-$\tau_1\tau_2$-closed if and only if $Cl(Int(A)) - A$ is wgI-$\tau_1\tau_2$-open.

**Proof.** Suppose that $A$ is a wgI-$\tau_1\tau_2$-closed set. Let $F$ be a $\tau_1\tau_2$-closed set and $F \subseteq \tau_1\tau_2$-$Cl(\tau_1\tau_2$-$Int(A)) - A$. By Theorem 4.2.6, we have $F \in I$ and there exists $U \in I$ such that $U = F$. Thus, $F - U \subseteq \tau_1\tau_2$-$Int\left[\tau_1\tau_2\text{-}Cl\left(\tau_1\tau_2\text{-}Cl\left(\tau_1\tau_2\text{-}Int(A)\right) - A\right)\right]$ and by Theorem 4.2.14, $\tau_1\tau_2$-$Cl(\tau_1\tau_2$-$Int(A)) - A$ is wgI-$\tau_1\tau_2$-open.

Conversely, suppose that $\tau_1\tau_2$-$Cl(\tau_1\tau_2$-$Int(A)) - A$ is wgI-$\tau_1\tau_2$-open. Let $G$ be a $\tau_1\tau_2$-open set and $A \subseteq G$. Then, we have

$$\begin{aligned}\left[\tau_1\tau_2\text{-}Cl\left(\tau_1\tau_2\text{-}Int(A)\right) \cap (X - G)\right] &\subseteq \left[\tau_1\tau_2\text{-}Cl\left(\tau_1\tau_2\text{-}Int(A)\right) \cap (X - A)\right] \\ &= \tau_1\tau_2\text{-}Cl\left(\tau_1\tau_2\text{-}Int(A)\right) - A.\end{aligned}$$

Since $\tau_1\tau_2$-$Cl(\tau_1\tau_2$-$Int(A)) \cap (X - G)$ is $\tau_1\tau_2$-closed and $\tau_1\tau_2$-$Cl(\tau_1\tau_2$-$Int(A)) - A$ is wgI-$\tau_1\tau_2$-open, by Theorem 4.2.11,

$$\left[\tau_1\tau_2\text{-}Cl\left(\tau_1\tau_2\text{-}Int(A)\right) \cap (X - G)\right] - U \subseteq \tau_1\tau_2\text{-}Cl\left(\tau_1\tau_2\text{-}Int(A)\right) - A \text{ for some } U \in I.$$

Since

$$
\begin{aligned}
&\left[\tau_1\tau_2\text{-}Cl\left(\tau_1\tau_2\text{-}Int\left(A\right)\right)\cap\left(X-G\right)\right]-U \\
&\subseteq \tau_1\tau_2\text{-}Int\left[\tau_1\tau_2\text{-}Cl\left[\tau_1\tau_2\text{-}Cl\left(\tau_1\tau_2\text{-}Int\left(A\right)\right)-A\right]\right] \\
&= \tau_1\tau_2\text{-}Int\left[\tau_1\tau_2\text{-}Cl\left[\tau_1\tau_2\text{-}Cl\left(\tau_1\tau_2\text{-}Int\left(A\right)\right)\cap\left(X-A\right)\right]\right] \\
&\subseteq \tau_1\tau_2\text{-}Int\left[\tau_1\tau_2\text{-}Cl\left(\tau_1\tau_2\text{-}Cl\left(\tau_1\tau_2\text{-}Int\left(A\right)\right)\cap\tau_1\tau_2\text{-}Cl\left(X-A\right)\right)\right] \\
&= \tau_1\tau_2\text{-}Int\left(\tau_1\tau_2\text{-}Cl\left(\tau_1\tau_2\text{-}Int\left(A\right)\right)\right)\cap\tau_1\tau_2\text{-}Int\left(\tau_1\tau_2\text{-}Cl\left(X-A\right)\right) \\
&= \tau_1\tau_2\text{-}Int\left(\tau_1\tau_2\text{-}Cl\left(\tau_1\tau_2\text{-}Int\left(A\right)\right)\right)\cap\left[X-\tau_1\tau_2\text{-}Cl\left(\tau_1\tau_2\text{-}Int\left(A\right)\right)\right] \\
&\subseteq \tau_1\tau_2\text{-}Cl\left(\tau_1\tau_2\text{-}Int\left(A\right)\right)\cap\left[X-\tau_1\tau_2\text{-}Cl\left(\tau_1\tau_2\text{-}Int\left(A\right)\right)\right]=\varnothing,
\end{aligned}
$$

$\tau_1\tau_2\text{-}Cl\left(\tau_1\tau_2\text{-}Int\left(A\right)\right)\cap\left(X-G\right)\subseteq U$ and hence $\tau_1\tau_2\text{-}Cl\left(\tau_1\tau_2\text{-}Int\left(A\right)\right)-G\in I$.

This show that $A$ is wgI-$\tau_1\tau_2$-closed. ❑

# CHAPTER 5

# CONCLUSIONS AND RECOMMENDATIONS

## 5.1 Conclusions

This research deals with the concept of weakly generalized closed sets in ideal topological spaces. Moreover, some properties of weakly generalized closed sets with respect to an ideal are investigated. Furthermore, several properties of generalized closed sets in ideal bitopological spaces are discussed. The results are as follows:

(I) For subsets $A$ and $B$ of an ideal topological space $(X,\tau,I)$, the following properties hold:

(1) $A$ is wgI-closed if and only if $F \subseteq Cl(Int(A)) - A$ and $F$ is closed in $X$ implies $F \in I$;

(2) $A$ is wgI-closed if and only if $F - U \subseteq Int(Cl(A))$ for some $U \in I$, whenever $F \subseteq A$ and $F$ is closed;

(3) If $A$ and $B$ are both wgI-closed and open sets, then $A \cup B$ is also wgI-closed;

(4) If $A$ is wgI-closed and $F$ is closed, then $A \cap F$ is also wgI-closed;

(5) If $A$ and $B$ are separated wgI-open sets, then $A \cup B$ is also wgI-open;

(6) If $A$ is wgI-closed and $A \subseteq B \subseteq Cl(Int(A))$, then $B$ is wgI-closed;

(7) If $A$ is wgI-open and $Int(Cl(A)) \subseteq B \subseteq A$, then $B$ is wgI-open;

(8) $A$ is wgI-closed if and only if $Cl(Int(A)) - A$ is wgI-open.

(II) For subsets $A$ and $B$ of an ideal bitopological space $(X,\tau_1,\tau_2,I)$, the following properties hold:

(1) $A$ is wgI-$\tau_1\tau_2$-closed if and only if $F \subseteq \tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Int(A)) - A$ and $F$ is $\tau_1\tau_2$-closed in $X$ implies $F \in I$;

(2) $A$ is wgI-$\tau_1\tau_2$-closed if and only if $F - U \subseteq \tau_1\tau_2\text{-}Int(\tau_1\tau_2\text{-}Cl(A)) - A$ for some $U \in I$, whenever $F \subseteq A$ and $F$ is $\tau_1\tau_2$-closed;

(3) If $A$ and $B$ are both wgI-$\tau_1\tau_2$-closed and $\tau_1\tau_2$-open sets, then $A \cup B$ is also wgI-$\tau_1\tau_2$-closed;

(4) If $A$ is wgI-$\tau_1\tau_2$-closed and $F$ is $\tau_1\tau_2$-closed, then $A \cap F$ is also wgI-$\tau_1\tau_2$-closed;

(5) If $A$ and $B$ are separated wgI-$\tau_1\tau_2$-open sets, then $A\cup B$ is also wgI-$\tau_1\tau_2$-open ;

(6) If $A$ is wgI-$\tau_1\tau_2$-closed and $A\subseteq B\subseteq \tau_1\tau_2\text{-}Cl\left(\tau_1\tau_2\text{-}Int\left(A\right)\right)$, then $B$ is wgI-$\tau_1\tau_2$-closed ;

(7) If $A$ is wgI-$\tau_1\tau_2$-open and $\tau_1\tau_2\text{-}Cl\left(\tau_1\tau_2\text{-}Int\left(A\right)\right)\subseteq B\subseteq A$, then $B$ is wgI-$\tau_1\tau_2$-open ;

(8) $A$ is wgI-$\tau_1\tau_2$-closed if and only if $\tau_1\tau_2\text{-}Cl\left(\tau_1\tau_2\text{-}Int\left(A\right)\right)-A$ is wgI-$\tau_1\tau_2$-open .

(III) For an ideal topological space $\left(X,\tau,I\right)$, the following properties are equivalent:

(1) $\left(X,\tau,I\right)$ is wgI-normal;

(2) for every wgI-closed set $F$ and wgI-open set $G$ containing $F$, there exists an open set $V$ such that $F-V\in I$ and $Cl\left(V\right)-G\in I$ ;

(3) for each pair of disjoint wgI-closed sets $A$ and $B$, there exists an open set $U$ such that $A-U\in I$ and $Cl\left(U\right)\cap B\in I$ .

(IV) For an ideal topological space $\left(X,\tau,I\right)$, the following properties are equivalent:

(1) $\left(X,\tau,I\right)$ is wgI-regular;

(2) for each $x\in X$ and wgI-open set $U$ containing $x$, there exists an open set $V$ containing $x$ such that $Cl\left(Int\left(V\right)\right)-U\in I$ ;

(3) for each $x\in X$ and wgI-closed sets $A$ not containing $x$, there exists an open set $V$ containing $x$ such that $Cl\left(Int\left(V\right)\right)\cap A\in I$ .

## 5.2 Recommendations

To this end, even though I have found several properties as presented in this thesis, there are several questions yet to be answered and it may be worth investigating in future studies. I formulate the questions as follows:

5.2.1 Are there properties of weakly generalized closed sets in ideal topological spaces.

5.2.2 Are there another condition for weakly generalized closed sets in ideal topological spaces.

5.2.3 Are there properties of weakly generalized $\tau_1\tau_2$ -closed sets in ideal bitopological spaces.

5.2.4 Are there another condition for weakly generalized $\tau_1\tau_2$ -closed sets in ideal bitopological spaces.

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